# The Optimal Surface Mitigated Multiple Targeting System
# Patent Application Technical Documentation

Timothy Hall

### Abstract

The purpose of the PQIC Optimal Surface Mitigated Multiple Targeting System (PQIC OSMMTS) is to define the equipment and processing necessary to produce, in real time, an error-bounded, self-monitoring and self-adjusting, likelihood-based Target Position Report for arbitrarily many self-identifying targets in a two-dimensional grid. Each target sends identifying information to an array of sensors strategically placed in its vicinity to maximize the likelihood that the system will calculate a position report as accurately and precisely as possible.

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## 1. Mission Statement

The purpose of the **PQIC**[TM] **Optimal Surface Mitigated Multiple Targeting System** (PQIC OS-MMTS[1]) is to define the equipment and processing necessary to produce, in real time, an error-bounded, self-monitoring and self-adjusting, likelihood-based Target Position Report for arbitrarily many self-identifying targets in a two-dimensional grid[2]. Each target sends identifying information to an array of sensors strategically placed in its vicinity to maximize the likelihood that the system will calculate a position report as accurately and precisely as possible. The OSMMTS uses analytical and ad-hoc mitigation and optimization techniques to reduce the error bounds on the target report to a practical minimum. This memorandum provides the baseline technical documentation of the analytical methods, construct guidelines, quantification methods, mitigation and optimization techniques, and programming details for implementing the PQIC OSMMTS. Additional supporting methods, documented separately, that supplement, complement, and refine the analytical methods found herein shall be issued in an ongoing series through PQIC technical documentation.

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[1] While the formal name of the system is PQIC OSMMTS for copyright, trademark, service mark, and patent purposes, the name most often used in this, and subsequent technical, marketing, and promotional documentation shall be the OSMMTS System, or simply the OSMMTS.

[2] A separate, yet related, targeting system for a three-dimensional grid will be covered in a separate set of PQIC documentation.

## 2.  The OSMMTS System Interface

The OSMMTS System Interface consists of one *Principal Application Specific Integrated Circuit* (PASIC) *Central Processing Unit* (CPU), generically referred to as the PCPU, a set (at least four) of remote sensing SDU's that send information to the PCPU, and a database of statically stored data that the PCPU accesses for parameter[3] data, algorithm exceptions, and other information, which are used to produce the Target Position Report (TPR)[4], as well as supporting reports as the implementation determines (see Figure 2). The PCPU, SDU's, and any database systems must be coordinated on and agree with an absolutely maintained time system, accurate to at least twice the precision of the anticipated Target Position Report. In this respect, the accuracy of the Target Position Report depends more on the maintenance of the time system than it does on the algorithmic introduction of error due to calculation round-off or lack of precision.

A Target Position Report is generated whenever a SDU sends a coordinated stream[5] of timing information to the PCPU. Since different SDU will send information at slightly different times about the same target, an absolute timing schedule must be used to ensure valid comparison of timing data from the SDU set.

A Target $\mathbf{T}$ may only initiate a signal to the SDU set when $t = 0 \mod \xi$, where $\xi = 10^n/\rho$ cycles in a $10^n$ Hz PCPU, where there are $\rho$ signals per second For example, for a 1 GHz PCPU, if a target sends a signal to the SDU set every $\frac{1}{2}$ second, then $\rho = 2$, and $\xi = \frac{10^9}{2} = \frac{10^9}{10^{\log_{10} 2}} = 10^{9-\log_{10} 2}$.

The *Effective Range* of the OSMMTS System is the maximum time for this receive/query/confirm period. It measures the farthest a target may be away from the closest qualifying set of SDU's and still be detected by the system.

A complete *Signal Period*, i.e., $\xi = 10^n/\rho$ cycles in a $10^n$ Hz PCPU, consists of six *Phases*, each encompassing an interaction between the PCPU, the SDU set, and the parameter database (see Figure 1).

1. *Receive*, during which the PCPU receives the detected signal information from the SDU set. This phase must last as long as the effective range, plus overhead time for communications between the SDU set and the PCPU. The information passed during this phase consists of:
   (a) SDU ID
   (b) Target ID
   (c) Time Of Signal Detection

   The SDU and Target ID are static codes used throughout all phases and signal periods. If either the SDU ID or the Target ID changes during a signal period, it must be through a formal change management process incorporated into the particular implementation of the OSMMTS System. It shall be the responsibility of the OSMMTS System implementation to ensure that changing SDU ID and/or Target ID are linked properly for inference purposes.

   The Time of Signal Detection is relative to the common absolute timing mechanisms in the OSMMTS System.

2. *Query*, during which the PCPU queries the sending SDU for a confirmation code to ensure communication integrity. If the confirmation code sent by the SDU is not correct (see the next phase), the PCPU queries the SDU again for the proper confirmation code. This is repeated up to a tunable number of iterations. If no correct confirmation code is received in the allotted time, the SDU is deactivated.

3. *Confirm*, during which the PCPU receives and processes the confirmation code sent by the SDU. It is during this phase that any required re-transmissions are also requested, received, and disposed.

4. *Process*, during which all calculations are completed to produce the Target Position Report, and subsequent reports for evaluation, quantification, and adjustment purposes.

5. *Report*, during which the Target Position Report and supporting information are made available on output channels, and during which any auxiliary communications with the SDU's is completed.

6. *Sync*, during which no processing activity is scheduled. This is useful when coordinated processing activities require synchronized signal periods.

   One signal period begins when the previous one ends. The sync phase may be used to coordinate any overhead processing issues to implement this requirement.

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[3] Parameter data consists of Demerit, History, and Confirmation values for each SDU. See Section 45 on page 38 for more information.

[4] See Section 42 on page 37.

[5] In the OSMMTS, "coordinated" means "timed to a cycle coordinated by a master clock."

# Signal Timing



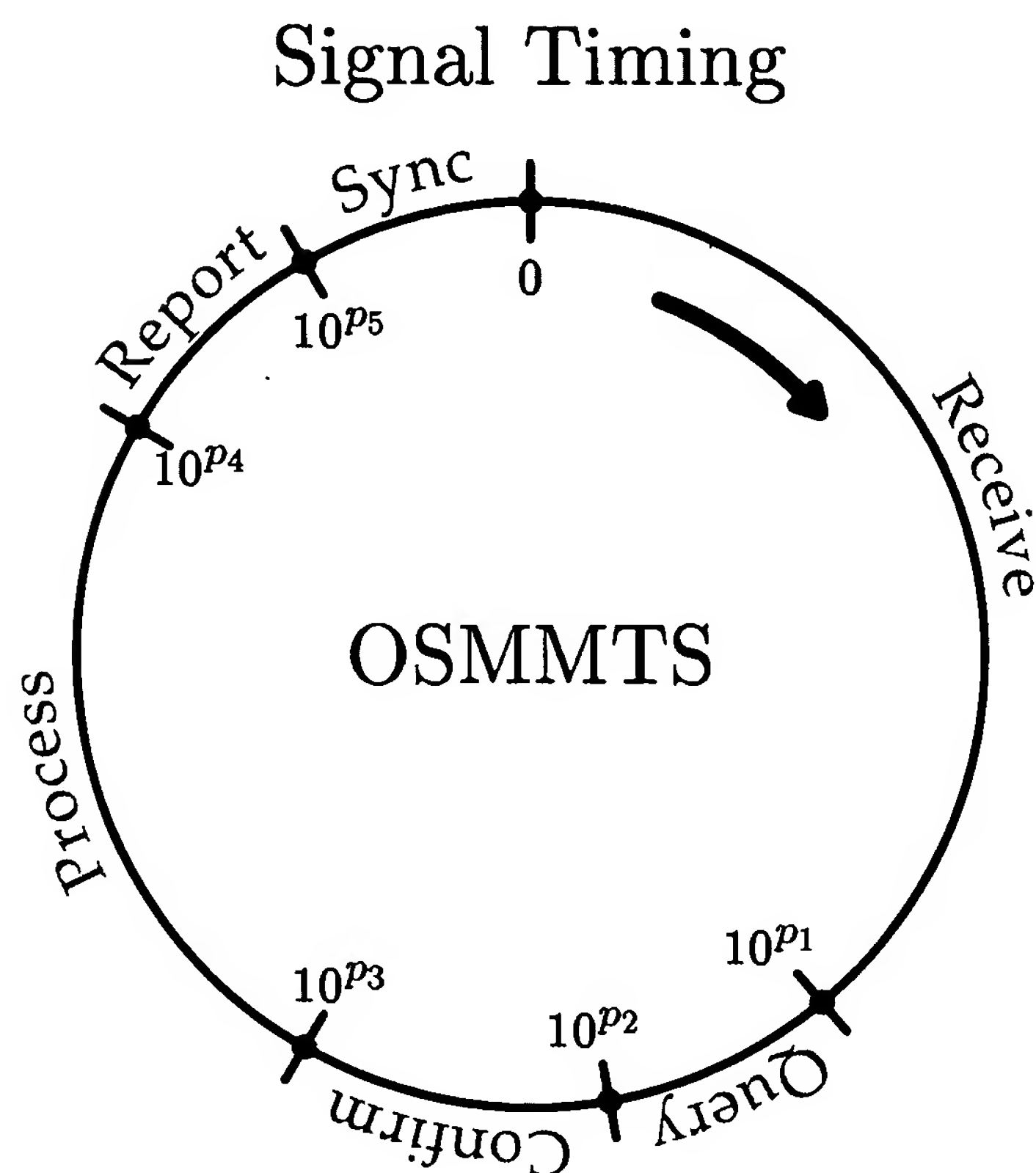**Figure 1**: The OSMMTS Signal Timing Cycle

A special SDU, the *Calibrating SDU* (CSDU), is placed where the PCPU can always receive its signal as clearly as possible, i.e., without corruption. This CSDU calibrates the system by being a known distance from all other SDU's, and allows for a consistent, guaranteed error-free TPR by which all other SDU's are evaluated for accuracy and consistency.
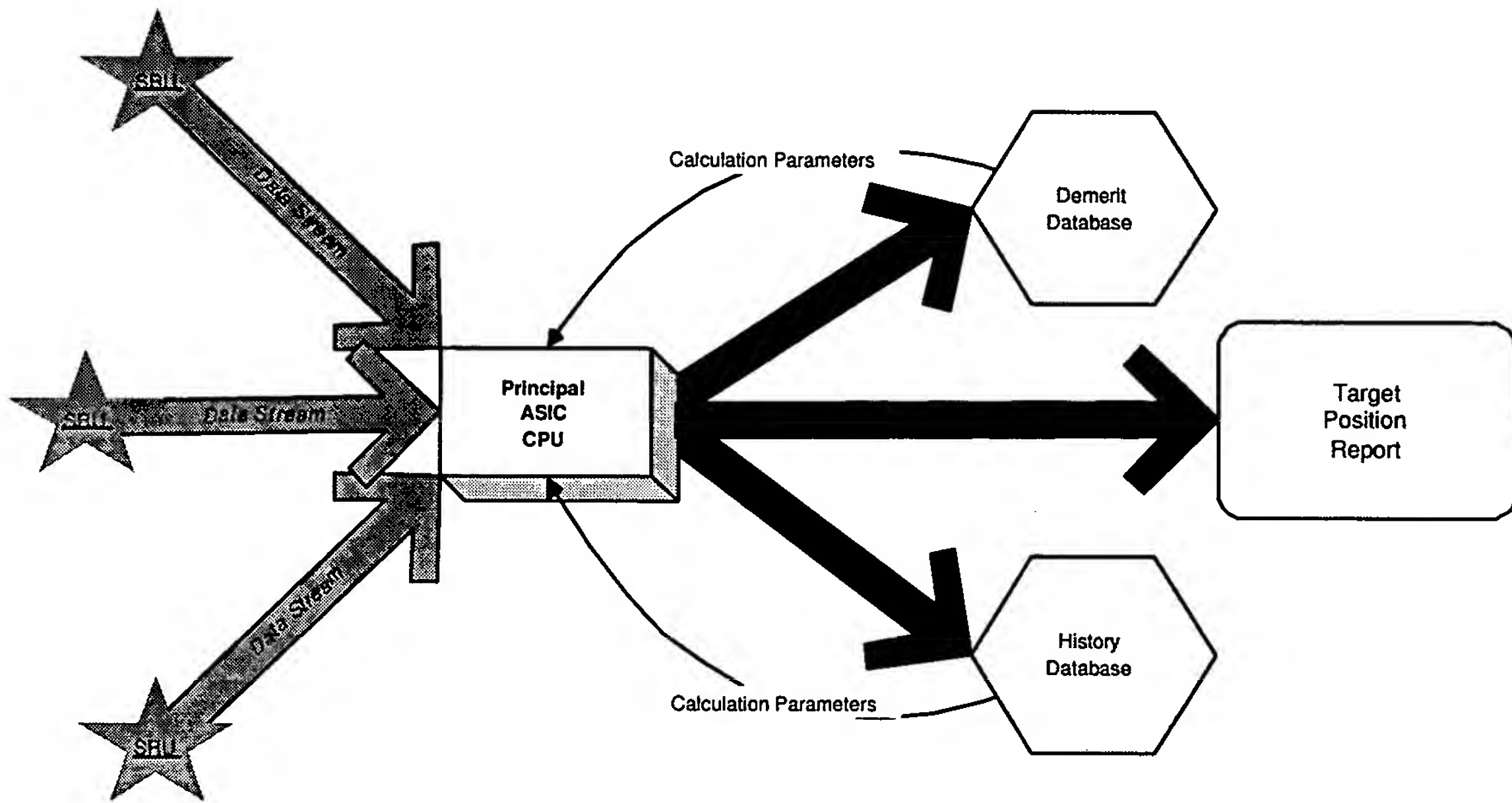
**Figure 2**: The OSMMTS Processing System

## 3.  Target Position Calculations

## 4.  Introduction

Suppose there were two Surface[6] Detection Units (SDU) labeled **I** and **II** that detect a signal that propagates through a uniformly dense environmental medium[7] encompassing the SDU and a Target **T**[8]. The signal from the target is detected at times $t_1$ and $t_2$, where the detection is made under a common methodology[9]. Suppose further that the SDU are a known distance $d > 0$ apart, and that **T** and the SDU are $s_1 > 0$ and $s_2 > 0$ distances apart, respectively. Finally, let $\theta$ be the angle (measured positively) between **T** and the SDU (see Figure 5). In the case that **T** is on the line segment between **I** and **II**, take $\theta = \frac{\pi}{2}$, i.e., the triangle formed by **I, II**, and **T** is degenerate, with two 0 angles between **I** and **T**, and **II** and **T**. If **I, II**, and **T** are otherwise collinear, then take $\theta = 0$; the other angles will be irrelevant.

The calculations that follow will depend on axioms that collectively assert the OSMMTS System. These axioms formally state the conditions or assumptions under which the OSMMTS System may be implemented.

**Axiom 1** Given $d$, and the positions of **I** and **II**, the only observed data, and therefore the only basis for any calculations found in the OSMMTS, are the times $t_1$ and $t_2$ of signal detection.

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[6] The detection units are called "surface" units because all coordinates under the OSMMTS may be thought of as $(x, y, z) \in \mathbb{R}^3$, were $z = 0$. Even after rotations and translations, the coordinates may be thought of as planes or surfaces embedded in $\mathbb{R}^3$.

[7] The term "environmental medium" is intentionally generic. The analytical and implementation aspects of the OSMMTS apply equally well to signals broadcast through the air, under water, in gas-filled containers, in vacuums, or in any other medium.

However, the medium will be assumed to be of uniform density to ensure the validity of the formula:

$$\text{Distance} = (\text{Linear Speed}) * \text{Time}$$

The "uniformly dense" restriction may be relaxed in subsequent OSMMTS System versions.

[8] Even though this development applies to a single Target **T**, all analytical methods in the OSMMTS may be extended to any number of individual targets with distinguishable signals.

[9] The "common methodology" refers to the electrical, mechanical, physical, or other well-defined, reproducible, and deterministic means of declaring a detection complete at an SDU. Whatever particular methods are used, the implementation must be uniform and constantly applied to each SDU in the OSMMTS System to ensure the integrity of algorithmic calculation.

## 5. Derivation

A critical element of the OSMMTS is to calculate a Target Position Report, which contains a position report based on the arrival times at the SDU's. Consider arrival times $t_1$ and $t_2$ at two SDU, named **I** and **II**, for the same Target **T**.
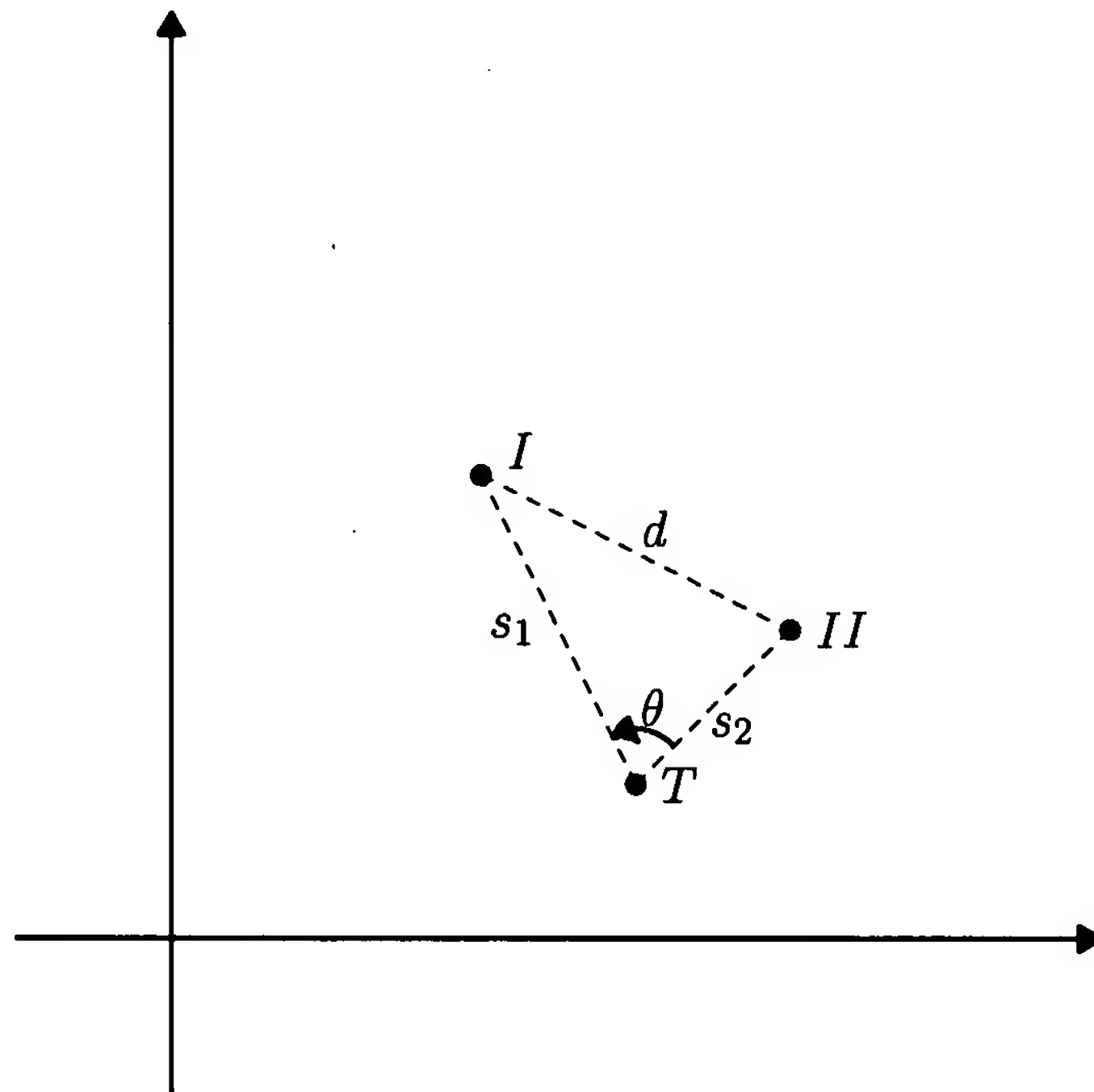


**Figure 3**: Relative placement of **I**, **II**, and **T**

For the triangle formed by **I**, **II**, and **T** depicted in Figure 5, the Law of Cosines gives

$$d^2 = s_1^2 + s_2^2 - 2s_1 s_2 \cos\theta \tag{1}$$

Let $f = t_1 - t_2$, $c =$ the speed with which the signal linearly propagates through the environmental medium, and let $t$ be the time at which the Target **T** sends the signal, with units in common. Note how $t_i = t$ only when **T** is identically collocated with either **I** or **II**. The term "speed" refers to the instantaneous change in distance per unit time, and it is expressed in terms of two-dimensional Euclidean geometry.

**Axiom 2** *For the purposes of the OSMMTS, it will be assumed, without loss of generality, that $t_i > t$, for $i = 1, 2$.*

Then

$$
\begin{aligned}
fc &= (t_1 - t_2)\,c \\
&= \left((t_1 - t) - (t_2 - t)\right)c \\
&= (t_1 - t)\,c - (t_2 - t)\,c \\
&= s_1 - s_2
\end{aligned}
$$

or

$$s_1^2 - 2s_1 s_2 + s_2^2 = (fc)^2$$

Hence,

$$
\begin{aligned}
s_1^2 + s_2^2 - 2s_1 s_2 \cos\theta - d^2 &= 0 \\
\underline{s_1^2 + s_2^2 - 2s_1 s_2 - (fc)^2 = 0} & \\
2s_1 s_2 \left(1 - \cos\theta\right) = d^2 - (fc)^2 &
\end{aligned}
$$

or

$$s_1 s_2 = \frac{d^2 - (fc)^2}{2\left(1 - \cos\theta\right)} = \kappa \tag{2}$$

since $\theta \neq 2n\pi$, for any integer $n$. If $\mathbf{I}$, $\mathbf{II}$, and $\mathbf{T}$ are collinear, e.g., the Target $\mathbf{T}$ is on the line between $\mathbf{I}$ and $\mathbf{II}$, then take $\theta = \pi$ (as in the case of a degenerate triangle). Assume otherwise[10] that $\mathbf{I}$, $\mathbf{II}$, and $\mathbf{T}$ are not collinear.

Note how

$$\kappa = \frac{d^2 - (fc)^2}{2(1 - \cos\theta)} \implies \cos\theta = 1 - \frac{d^2 - (fc)^2}{2\kappa}, \ \kappa \neq 0 \tag{3}$$

Here,

$$\begin{cases} \kappa > 0, & \text{when } d > |f|\, c \\ \kappa = 0, & \text{when } d = |f|\, c \\ \kappa < 0, & \text{when } d < |f|\, c \end{cases}$$

since $d$ and $c$ are both necessarily positive.

Now $s_1 - s_2 = fc$ gives us

$$s_1(s_1 - fc) = \kappa$$

$$s_1^2 - fcs_1 - \kappa = 0$$

$$\left(s_1 - \frac{fc}{2}\right)^2 = \kappa + \left(\frac{fc}{2}\right)^2$$

$$s_1 = \frac{fc}{2} \pm \sqrt{\kappa + \left(\frac{fc}{2}\right)^2}$$

Hence,

$$s_1 = \frac{1}{2}\left(fc \pm \sqrt{4\kappa + (fc)^2}\right) \text{ and } s_2 = \frac{2\kappa}{fc \pm \sqrt{4\kappa + (fc)^2}} \tag{4}$$

The choice of sign in (4) depends on the signs of $f$ and $\kappa$; whichever makes $s_1 > 0$ *and* $s_2 > 0$ should be used.

However, since $s_1 > 0$ and $s_2 > 0$, then $\kappa \neq 0$, hence

$$\begin{aligned} \frac{2\kappa}{fc \pm \sqrt{4\kappa + (fc)^2}} &= \left(\frac{2\kappa}{fc \pm \sqrt{4\kappa + (fc)^2}}\right)\left(\frac{fc \mp \sqrt{4\kappa + (fc)^2}}{fc \mp \sqrt{4\kappa + (fc)^2}}\right) \\ &= \left(\frac{2\kappa}{fc \pm \sqrt{4\kappa + (fc)^2}}\right)\left(\frac{fc \mp \sqrt{4\kappa + (fc)^2}}{fc \mp \sqrt{4\kappa + (fc)^2}}\right) \\ &= \frac{2\kappa\left(fc \mp \sqrt{4\kappa + (fc)^2}\right)}{-4\kappa} \\ &= -\frac{\left(fc \mp \sqrt{4\kappa + (fc)^2}\right)}{2} \\ &= \tfrac{1}{2}\left(-fc \pm \sqrt{4\kappa + (fc)^2}\right) \end{aligned}$$

So

$$s_1 = \frac{1}{2}\left(fc \pm \sqrt{4\kappa + (fc)^2}\right) \text{ and } s_2 = \frac{1}{2}\left(-fc \pm \sqrt{4\kappa + (fc)^2}\right) \tag{5}$$

Expression (5) shows that the $s_i$ are functions of $\kappa$, $f$, and $c$. Since $\kappa$ is a function of $d$, $f$, $c$, and $\theta$, we have the $s_i$ as functions of $d$, $f$, $c$, and $\theta$. The values of $d$ and $c$ are fixed by the circumstances and by nature, and the $f$ value is observed; this means the $s_i$ may be thought of as functions of $\theta$ given $d$ and the data $t_1$ and $t_2$.

We have

$$\begin{aligned} 4\kappa + (fc)^2 &= 4\left(\frac{d^2 - (fc)^2}{2(1 - \cos\theta)}\right) + (fc)^2 \\ &= 2\left(\frac{d^2 - (fc)^2}{1 - \cos\theta}\right) + \frac{(1 - \cos\theta)(fc)^2}{1 - \cos\theta} \\ &= \frac{2d^2 - 2(fc)^2 + (1 - \cos\theta)(fc)^2}{1 - \cos\theta} \\ &= \frac{2d^2 - (1 + \cos\theta)(fc)^2}{1 - \cos\theta} \end{aligned}$$

hence

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[10] If $\mathbf{I}$, $\mathbf{II}$, and $\mathbf{T}$ are collinear, yet the Target $\mathbf{T}$ is not on the line between $\mathbf{I}$ and $\mathbf{II}$, then while it would make sense to take $\theta = 0$, this would make $\kappa$ ill-defined. For the purposes of the OSMMTS development, assume $\mathbf{I}$, $\mathbf{II}$, and $\mathbf{T}$ are collinear *only* when the Target $\mathbf{T}$ is on the line between $\mathbf{I}$ and $\mathbf{II}$.

$$s_1\left(\theta|d, \{t_1, t_2\}\right) = \tfrac{1}{2}\left((t_1 - t_2)c \pm \sqrt{\tfrac{2d^2 - (1+\cos\theta)((t_1-t_2)c)^2}{(1-\cos\theta)}}\right) > 0$$

$$s_2\left(\theta|d, \{t_1, t_2\}\right) = \tfrac{1}{2}\left((t_2 - t_1)c \pm \sqrt{\tfrac{2d^2 - (1+\cos\theta)((t_1-t_2)c)^2}{(1-\cos\theta)}}\right) > 0$$

Note how

$$
\begin{aligned}
s_1^2 - s_2^2 &= \tfrac{1}{4}\left(fc \pm \sqrt{4\kappa + (fc)^2}\right)^2 - \tfrac{1}{4}\left(-fc \pm \sqrt{4\kappa + (fc)^2}\right)^2 \\
&= \tfrac{1}{4}\left(\begin{array}{c}(fc)^2 \pm 2fc\sqrt{4\kappa + (fc)^2} + 4\kappa + (fc)^2 \\ -(-fc)^2 \pm 2fc\sqrt{4\kappa + (fc)^2} - 4\kappa - (fc)^2\end{array}\right) \\
&= \pm fc\sqrt{4\kappa + (fc)^2}
\end{aligned}
\tag{6}
$$

Hence,

$$
\begin{aligned}
s_1 + s_2 &= \tfrac{s_1^2 - s_2^2}{s_1 - s_2} \\
&= \tfrac{\pm fc\sqrt{4\kappa + (fc)^2}}{fc} \\
&= \pm\sqrt{4\kappa + (fc)^2}
\end{aligned}
$$
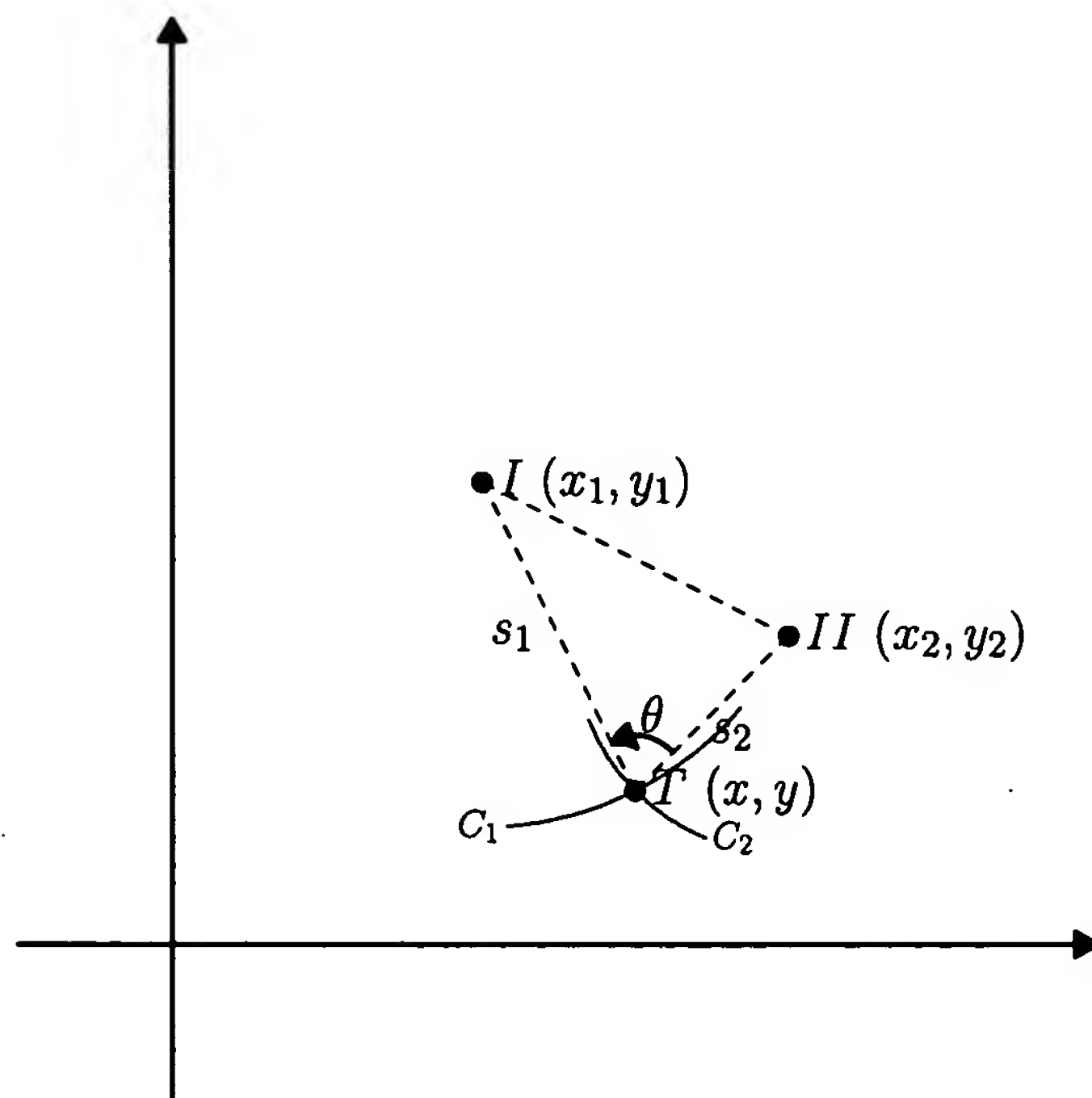


**Figure 4:** Intersection of Circles $C_1$ and $C_2$

Since $s_1$ and $s_2$ are distances from **T** to **I** and **II**, respectively, then **T** must be at the intersection of two circles centered at **I** and **II** (see Figure 4). In coordinate geometry, suppose **I** is at $(x_1, y_1)$, **II** is at $(x_2, y_2)$, and **T** is at $(x, y)$. The origin for the coordinate system is arbitrary, and its position may be chosen for convenience in any particular implementation. Then

$$(x - x_1)^2 + (y - y_1)^2 = s_1^2$$
$$(x - x_2)^2 + (y - y_2)^2 = s_2^2$$

must be satisfied simultaneously for a given $\theta$. Hence, from (6),

$$x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 = s_1^2$$
$$x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2 = s_2^2$$
$$\overline{2x(x_2 - x_1) + 2y(y_2 - y_1) = \left(s_1^2 - s_2^2\right) + \left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right)}$$
$$2x(x_2 - x_1) + 2y(y_2 - y_1) = \pm fc\sqrt{4\kappa + (fc)^2} + \left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right)$$

Now if $y_1 = y_2$, then we have

$$2x(x_2 - x_1) = \pm fc\sqrt{4\kappa + (fc)^2} + \left(x_2^2 - x_1^2\right)$$

or

$$x = \frac{\pm fc\sqrt{4\kappa + (fc)^2} + \left(x_2^2 - x_1^2\right)}{2(x_2 - x_1)}$$

which is a vertical line. If $y_1 \neq y_2$, then we have

$$2x(x_2 - x_1) + 2y(y_2 - y_1) = \pm fc\sqrt{4\kappa + (fc)^2} + \left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right)$$

or

$$y = -x\left(\frac{x_2 - x_1}{y_2 - y_1}\right) + \left[\frac{\pm fc\sqrt{4\kappa + (fc)^2} + \left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right)}{2(y_2 - y_1)}\right]$$

which is a line with slope $-\left(\frac{x_2-x_1}{y_2-y_1}\right)$ and $y$-intercept $\frac{\pm fc\sqrt{4\kappa+(fc)^2}+\left(x_2^2-x_1^2\right)+\left(y_2^2-y_1^2\right)}{2(y_2-y_1)}$. To summarize, let

$$L_1 : \begin{cases} x = \frac{\pm fc\sqrt{4\kappa+(fc)^2}+\left(x_2^2-x_1^2\right)}{2(x_2-x_1)}, & y_1 = y_2 \\ y = -x\left(\frac{x_2-x_1}{y_2-y_1}\right) + \left[\frac{\pm fc\sqrt{4\kappa+(fc)^2}+\left(x_2^2-x_1^2\right)+\left(y_2^2-y_1^2\right)}{2(y_2-y_1)}\right], & y_1 \neq y_2 \end{cases} \tag{7}$$

is a line in $x$ and $y$. Hence, as $\theta$ varies, the set of points of intersection between Circles **I** and **II** is a line defined by $L_1$ (see Figure 5).
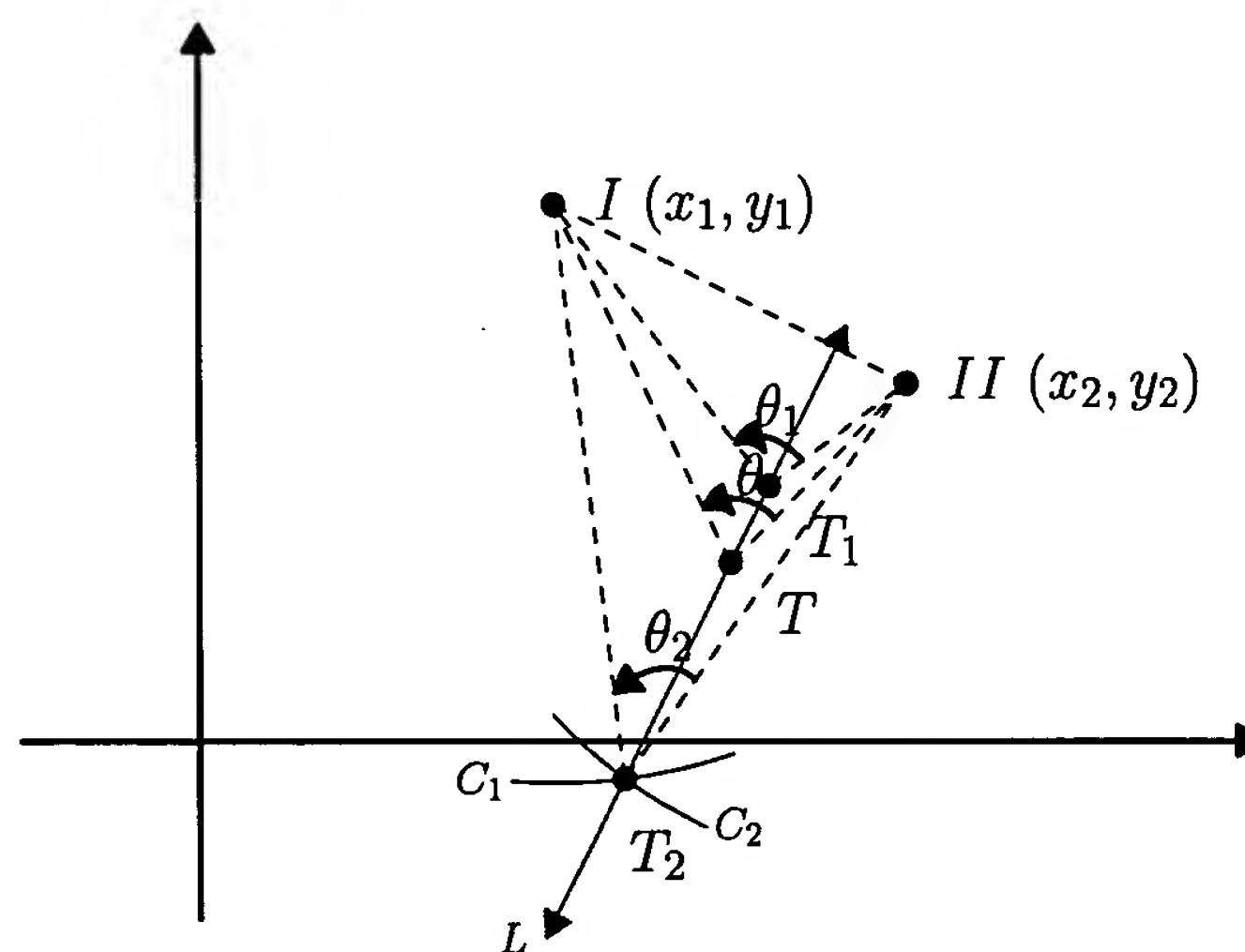


Figure 5: Foci of Points as $\theta$ Varies

If a third SDU, named **III**, is added to the circumstance, at position $(x_3, y_3)$, with sides relabeled to account for the third SDU, and which is not collinear to the positions of SDU **I** and **II** (see Figure 6), and subject to the axioms mentioned earlier, e.g., see Axiom 2, then another line of points of intersection $L_2$, defined as in (7), may be expressed in terms of the same equations as before, this time for Circles **I** and **III** (see Figure 7). In particular, the equation for $L_1$ becomes

$$L_1 : \begin{cases} x = \frac{\pm f_1 c\sqrt{4\kappa_1+(f_1 c)^2}+\left(x_2^2-x_1^2\right)}{2(x_2-x_1)}, & y_1 = y_2 \\ y = -x\left(\frac{x_2-x_1}{y_2-y_1}\right) + \left[\frac{\pm f_1 c\sqrt{4\kappa_1+(f_1 c)^2}+\left(x_2^2-x_1^2\right)+\left(y_2^2-y_1^2\right)}{2(y_2-y_1)}\right], & y_1 \neq y_2 \end{cases}$$

and

$$L_2 : \begin{cases} x = \dfrac{\pm f_2 c\sqrt{4\kappa_2+(f_2 c)^2}+\left(x_3^2-x_1^2\right)}{2(x_3-x_1)}, & y_1 = y_3 \\[2ex] y = -x\left(\dfrac{x_3-x_1}{y_3-y_1}\right) + \left[\dfrac{\pm f_2 c\sqrt{4\kappa_2+(f_2 c)^2}+\left(x_3^2-x_1^2\right)+\left(y_3^2-y_1^2\right)}{2(y_3-y_1)}\right], & y_1 \neq y_3 \end{cases}$$

for distinct $\kappa_1$ and $\kappa_2$, themselves functions of distinct $\theta_1$ and $\theta_2$.
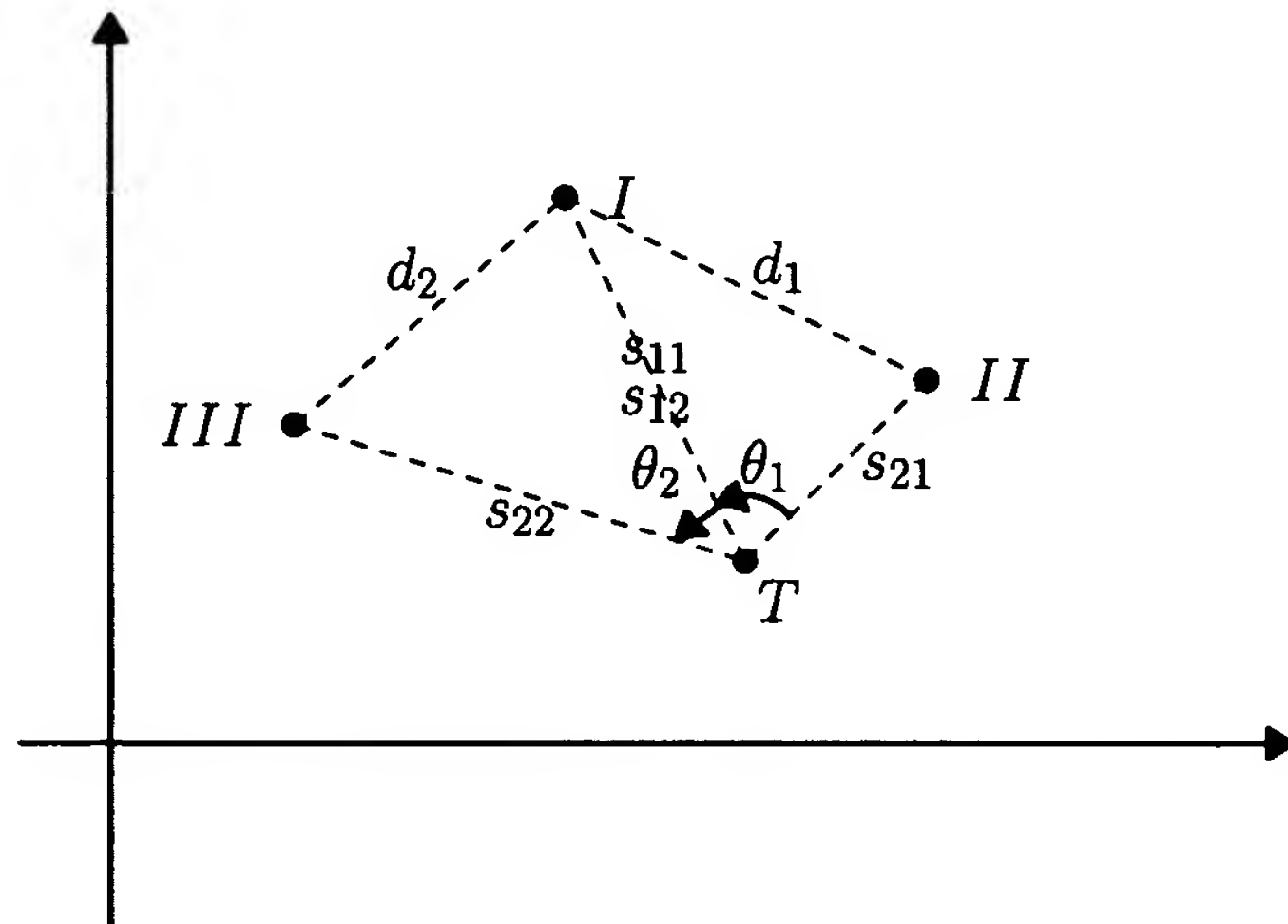


**Figure 6**: Third SDU Added

However, since $s_{11} = s_{12}$ (see the common side in Figure 6), then by (5), we have a link between $\kappa_1$ and $\kappa_2$, namely

$$\frac{1}{2}\left(f_1 c \pm \sqrt{4\kappa_1 + (f_1 c)^2}\right)$$

$$= \frac{1}{2}\left(-f_2 c \pm \sqrt{4\kappa_2 + (f_2 c)^2}\right)$$

or

$$(f_1 + f_2)\, c \pm \sqrt{4\kappa_1 + (f_1 c)^2} = \pm\sqrt{4\kappa_2 + (f_2 c)^2}.$$

Therefore, we have

$$L_1 : \begin{cases} x = \dfrac{\pm f_1 c\sqrt{4\kappa_1+(f_1 c)^2}+\left(x_2^2-x_1^2\right)}{2(x_2-x_1)}, & y_1 = y_2 \\[2ex] y = -x\left(\dfrac{x_2-x_1}{y_2-y_1}\right) + \left[\dfrac{\pm f_1 c\sqrt{4\kappa_1+(f_1 c)^2}+\left(x_2^2-x_1^2\right)+\left(y_2^2-y_1^2\right)}{2(y_2-y_1)}\right], & y_1 \neq y_2 \end{cases} \qquad (8a)$$

and

$$L_2 : \begin{cases} x = \dfrac{f_2 c\left((f_1+f_2)c\pm\sqrt{4\kappa_1+(f_1 c)^2}\right)+\left(x_3^2-x_1^2\right)}{2(x_3-x_1)}, & y_1 = y_3 \\[2ex] y = -x\left(\dfrac{x_3-x_1}{y_3-y_1}\right) + \left[\dfrac{f_2 c\left((f_1+f_2)c\pm\sqrt{4\kappa_1+(f_1 c)^2}\right)+\left(x_3^2-x_1^2\right)+\left(y_3^2-y_1^2\right)}{2(y_3-y_1)}\right], & y_1 \neq y_3 \end{cases} \qquad (9a)$$

**Claim 3** *If I, II, and III are not collinear, then $L_1$ and $L_2$ are not parallel.*

*Proof.* Let $\mu_{ij}$ be the slope of the line between SDU $i$ and $j$, e.g., $\mu_{23} = \frac{y_2-y_3}{x_2-x_3}$, or $\mu_{23} = +\infty$ for a vertical line. Since **I**, **II**, and **III** are not collinear, we have

$$\mu_{12} \neq \mu_{23} \neq \mu_{13}$$

with allowances for dealing with infinities, and $x_1, x_2, x_3$ are not all the same value.

Suppose $y_1 \neq y_2$ and $y_1 \neq y_3$. Then the slope of $L_1$ is $-\left(\frac{x_2-x_1}{y_2-y_1}\right)$, and for $L_2$ it is $-\left(\frac{x_3-x_1}{y_3-y_1}\right)$. If

$$-\left(\frac{x_2-x_1}{y_2-y_1}\right) = -\left(\frac{x_3-x_1}{y_3-y_1}\right)$$
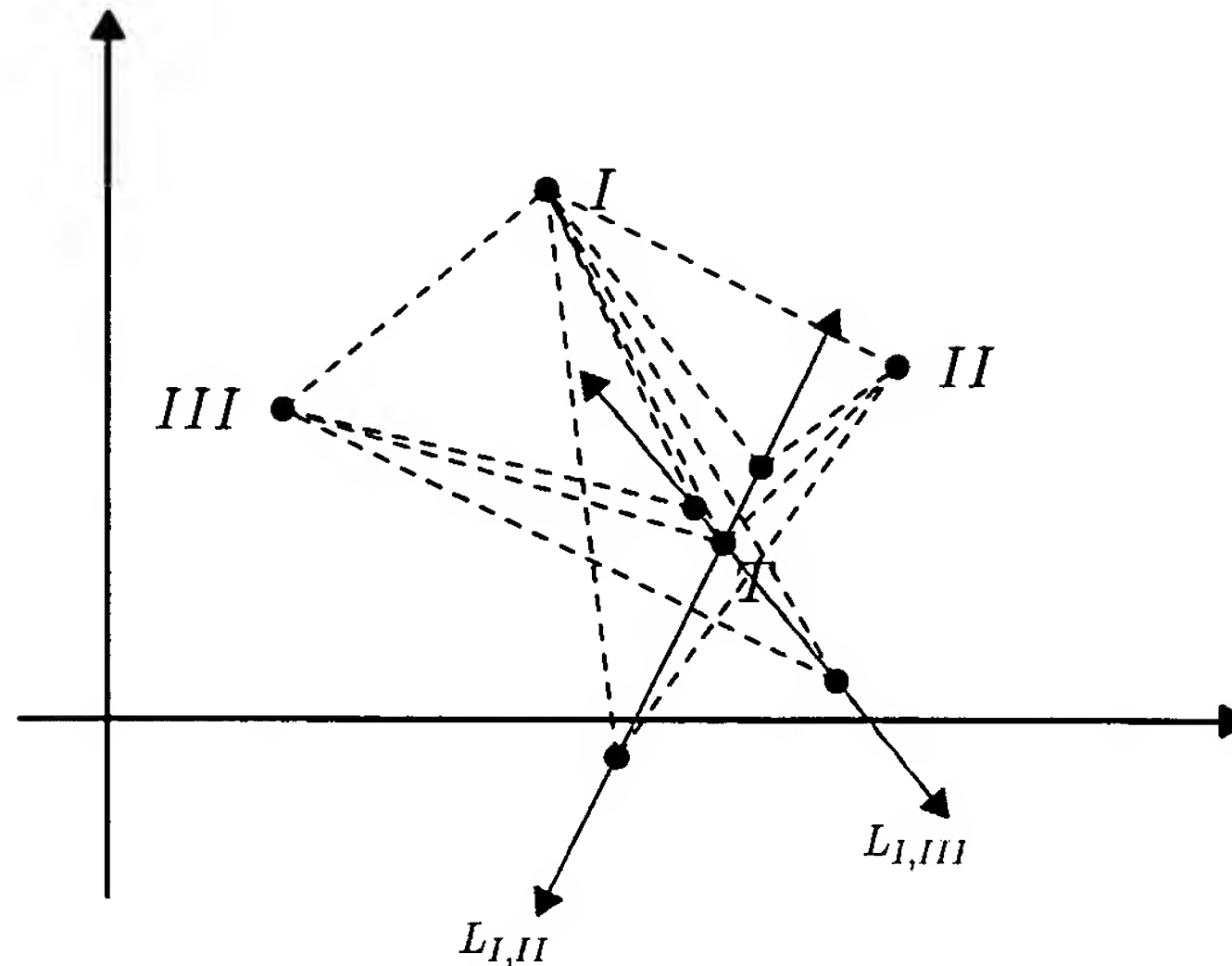
**Figure 7**: Intersection of Foci Locates **T**

then either

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1}$$

or

$$+\infty = \frac{y_3 - y_1}{x_3 - x_1}, \text{ when } x_1 = x_2$$

or

$$\frac{y_2 - y_1}{x_2 - x_1} = +\infty, \text{ when } x_1 = x_3$$

If $\frac{y_2-y_1}{x_2-x_1} = \frac{y_3-y_1}{x_3-x_1}$, then $\mu_{12} = \mu_{13}$; a contradiction.

If $+\infty = \frac{y_3-y_1}{x_3-x_1}$, then $x_1 = x_3$, since $y_1 \neq y_3$. This is also a contradiction, since $x_1 = x_2$ under this situation.

If $\frac{y_2-y_1}{x_2-x_1} = +\infty$, then $x_1 = x_2$, since $y_1 \neq y_2$, yet another contradiction, since $x_1 = x_3$ under this situation.

If either $y_1 = y_2$ and $y_2 \neq y_3$, or $y_2 = y_3$ and $y_1 \neq y_2$, then the slope of $L_1$ would be $+\infty$, and for $L_2$ it would be $-\left(\frac{x_3-x_1}{y_3-y_1}\right)$, or the other way around. In either case, these slopes could not be equal in any sense.

Finally, if both $y_1 = y_2$ and $y_1 = y_3$, then **I**, **II**, and **III** would be collinear, since they would all have the same $y$-value; this is also a contradiction. ■

By Claim 3, let $\theta_0$ be the value of $\theta$ for the point of intersection between $L_1$ and $L_2$. When $y_1 \neq y_2$ and $y_1 \neq y_3$, let

$$m_1 = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right) \text{ and } m_2 = -\left(\frac{x_3 - x_1}{y_3 - y_1}\right)$$

with

$$b_1 = \frac{\pm f_1 c\sqrt{4\kappa_0 + (f_1 c)^2} + \left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right)}{2\left(y_2 - y_1\right)}$$

$$b_2 = \frac{\pm f_2 c\sqrt{4\kappa_0 + (f_2 c)^2} + \left(x_3^2 - x_1^2\right) + \left(y_3^2 - y_1^2\right)}{2\left(y_3 - y_1\right)}$$

for $\kappa_0$ corresponding to $\theta_0$.

Then

$$L_1 : y = m_1 x + b_1$$
$$L_2 : y = m_2 x + b_2$$

**Claim 4** *When I, II, and III are not collinear, and $y_1 \neq y_2$ and $y_1 \neq y_3$, the point of intersection between $L_1$ and $L_2$ is $(x_0, y_0) = \left( \frac{b_2 - b_1}{m_1 - m_2}, m_1 \left( \frac{b_2 - b_1}{m_1 - m_2} \right) + b_1 \right)$.*

*Proof.* We have

$$m_1 x_0 + b_1 = m_2 x_0 + b_2$$

or

$$x_0 = \frac{b_2 - b_1}{m_1 - m_2}$$

and

$$
\begin{aligned}
y_0|_{L_1} &= m_1 x_0 + b_1 \\
&= m_1 \left( \frac{b_2 - b_1}{m_1 - m_2} \right) + b_1 \\
&= (m_1 - m_2 + m_2) \left( \frac{b_2 - b_1}{m_1 - m_2} \right) + b_1 \\
&= b_2 - b_1 + b_1 + m_2 \left( \frac{b_2 - b_1}{m_1 - m_2} \right) \\
&= m_2 \left( \frac{b_2 - b_1}{m_1 - m_2} \right) + b_2 \\
&= y_0|_{L_2}
\end{aligned}
$$

∎

**Claim 5** *When I, II, and III are not collinear, and $y_1 = y_2$ and $y_1 \neq y_3$, the point of intersection between $L_1$ and $L_2$ is*

$$
(x_0, y_0) = \left( \frac{\pm f_1 c \sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2)}{2(x_2 - x_1)}, \frac{1}{2(y_3 - y_1)} \left( \begin{array}{l} f_2 c \left( (f_1 + f_2) c \pm \sqrt{4\kappa_0 + (f_1 c)^2} \right) + (x_3^2 - x_1^2) + (y_3^2 - y_1^2) \\ - \left( \pm f_1 c \sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2) \right) \left( \frac{x_3 - x_1}{x_2 - x_1} \right) \end{array} \right) \right)
$$

*Proof.* We have

$$L_1 : x = \frac{\pm f_1 c \sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2)}{2(x_2 - x_1)}$$

and

$$L_2 : y = -x \left( \frac{x_3 - x_1}{y_3 - y_1} \right) + \left[ \frac{f_2 c \left( (f_1 + f_2) c + \sqrt{4\kappa_0 + (f_1 c)^2} \right) + (x_3^2 - x_1^2) + (y_3^2 - y_1^2)}{2(y_3 - y_1)} \right]$$

Hence

$$
\begin{aligned}
y_0|_{L_2} &= -x_0 \left( \frac{x_3 - x_1}{y_3 - y_1} \right) + \left[ \frac{f_2 c \left( (f_1 + f_2) c \pm \sqrt{4\kappa_0 + (f_1 c)^2} \right) + (x_3^2 - x_1^2) + (y_3^2 - y_1^2)}{2(y_3 - y_1)} \right] \\
&= - \left( \frac{\pm f_1 c \sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2)}{2(x_2 - x_1)} \right) \left( \frac{x_3 - x_1}{y_3 - y_1} \right) + \left[ \frac{f_2 c \left( (f_1 + f_2) c \pm \sqrt{4\kappa_0 + (f_1 c)^2} \right) + (x_3^2 - x_1^2) + (y_3^2 - y_1^2)}{2(y_3 - y_1)} \right] \\
&= - \left( \frac{\pm f_1 c \sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2)}{2(y_3 - y_1)} \right) \left( \frac{x_3 - x_1}{x_2 - x_1} \right) + \left[ \frac{f_2 c \left( (f_1 + f_2) c \pm \sqrt{4\kappa_0 + (f_1 c)^2} \right) + (x_3^2 - x_1^2) + (y_3^2 - y_1^2)}{2(y_3 - y_1)} \right] \\
&= \frac{1}{2(y_3 - y_1)} \left( \begin{array}{l} f_2 c \left( (f_1 + f_2) c \pm \sqrt{4\kappa_0 + (f_1 c)^2} \right) + (x_3^2 - x_1^2) + (y_3^2 - y_1^2) \\ - \left( \pm f_1 c \sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2) \right) \left( \frac{x_3 - x_1}{x_2 - x_1} \right) \end{array} \right)
\end{aligned}
$$

∎

**Claim 6** *When I, II, and III are not collinear, and $y_1 \neq y_2$ and $y_1 = y_3$, the point of intersection between $L_1$ and $L_2$ is*

$$
(x_0, y_0) = \left( \frac{f_2 c^2 (f_1 + f_2) \pm \sqrt{4\kappa_0 + (f_1 c)^2} + (x_3^2 - x_1^2)}{2(x_3 - x_1)}, \frac{1}{2(y_2 - y_1)} \left( \begin{array}{l} \pm f_1 c \sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2) + (y_2^2 - y_1^2) \\ - \left( f_2 c \left( (f_1 + f_2) c \pm \sqrt{4\kappa_0 + (f_1 c)^2} \right) + (x_3^2 - x_1^2) \right) \left( \frac{x_2 - x_1}{x_3 - x_1} \right) \end{array} \right) \right)
$$

*Proof.* We have

$$L_1 : y = -x\left(\frac{x_2 - x_1}{y_2 - y_1}\right) + \left[\frac{\pm f_1 c\sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2) + (y_2^2 - y_1^2)}{2(y_2 - y_1)}\right]$$

and

$$L_2 : x = \frac{f_2 c\left((f_1 + f_2)c \pm \sqrt{4\kappa_0 + (f_1 c)^2}\right) + (x_3^2 - x_1^2)}{2(x_3 - x_1)}$$

Hence

$$
\begin{aligned}
y_0|_{L_1} &= -x_0\left(\frac{x_2 - x_1}{y_2 - y_1}\right) + \left[\frac{\pm f_1 c\sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2) + (y_2^2 - y_1^2)}{2(y_2 - y_1)}\right] \\
&= -\left(\frac{f_2 c\left((f_1 + f_2)c \pm \sqrt{4\kappa_0 + (f_1 c)^2}\right) + (x_3^2 - x_1^2)}{2(x_3 - x_1)}\right)\left(\frac{x_2 - x_1}{y_2 - y_1}\right) + \left[\frac{\pm f_1 c\sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2) + (y_2^2 - y_1^2)}{2(y_2 - y_1)}\right] \\
&= -\left(\frac{f_2 c\left((f_1 + f_2)c \pm \sqrt{4\kappa_0 + (f_1 c)^2}\right) + (x_3^2 - x_1^2)}{2(y_2 - y_1)}\right)\left(\frac{x_2 - x_1}{x_3 - x_1}\right) + \left[\frac{\pm f_1 c\sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2) + (y_2^2 - y_1^2)}{2(y_2 - y_1)}\right] \\
&= \frac{1}{2(y_2 - y_1)}\left(\begin{array}{c} \pm f_1 c\sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2) + (y_2^2 - y_1^2) \\ -\left(f_2 c\left((f_1 + f_2)c \pm \sqrt{4\kappa_0 + (f_1 c)^2}\right) + (x_3^2 - x_1^2)\right)\left(\frac{x_2 - x_1}{x_3 - x_1}\right) \end{array}\right)
\end{aligned}
$$

∎

The fourth possibility consistent with Claims 4-6, namely $y_1 = y_2$ and $y_1 = y_3$, has been excluded by the assumption that **I**, **II**, and **III** are not collinear.

Note how in all Claims 4-6, $(x_0, y_0)$ is a function of constants and the $s_{ij}$ (via the $b_i$), which are in turn functions of $\theta$. This means $\theta_0$ may be expressed implicitly through the point $(x_0, y_0)$.

If

$$V_1 = (x_1, y_1) - (x_0, y_0)$$

and

$$V_2 = (x_2, y_2) - (x_0, y_0)$$

then $|V_1|_2 = s_{11}$ and $|V_2|_2 = s_{21}$, so that

$$V_1 \cdot V_2 = |V_1|_2 |V_2|_2 \cos\theta_0 = s_{11}s_{21}\cos\theta_0$$

This proves the following claim.

**Claim 7** $\theta_0 = Arc\cos\left(\frac{V_1 \cdot V_2}{s_{11}s_{21}}\right)$

The quadrant of $\theta_0$ must be considered when calculating the inverse cosine in Claim 7. Once established, we then have

$$s_{11}(\theta_0) s_{21}(\theta_0)\cos\theta_0 = (x_1 - x_0(\theta_0))(x_2 - x_0(\theta_0)) + (y_1 - y_0(\theta_0))(y_2 - y_0(\theta_0)) \tag{10}$$

However, from (2),

$$s_{11}(\theta_0) s_{21}(\theta_0) = \kappa_0$$

and, from (3),

$$\cos\theta_0 = 1 - \frac{d_1^2 - (f_1 c)^2}{2\kappa_0}$$

so

$$s_{11}(\theta_0) s_{21}(\theta_0)\cos\theta_0 = \kappa_0\left(1 - \frac{d_1^2 - (f_1 c)^2}{2\kappa_0}\right) \tag{11}$$

or

$$\kappa_0 = \frac{d_1^2 - (f_1 c)^2}{2} + (x_1 - x_0(\theta_0))(x_2 - x_0(\theta_0)) + (y_1 - y_0(\theta_0))(y_2 - y_0(\theta_0))$$

by (10).

Solving for $\kappa_0$, and substituting this value into Claim 4, 5, or 6, depending on circumstances, determines the position of Target **T**. In fact,

$$t = t_1 - \frac{s_1}{c} = t_1 - \frac{1}{2c}\left(f_1 c \pm \sqrt{4\kappa_0 + (f_1 c)^2}\right)$$

## 6. Calculation Algorithm

To summarize, given SDU positions $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$, and observed data $\{t_1, t_2, t_3\}$, the following algorithms give the Calculated Target Position $(x_0, y_0)$ of Target $\mathbf{T}$.

## 7. Common Elements

A. Calculate $f_1 = t_1 - t_2$ and $f_2 = t_1 - t_3$.

B. Calculate $d_1 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $d_2 = \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}$.

## 8. When $y_1 \neq y_2$, $y_1 \neq y_3$, and $d_i > |f_i|\, c$, for $i = 1, 2$

1. Calculate $m_1 = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)$.

2. Calculate $m_2 = -\left(\frac{x_3 - x_1}{y_3 - y_1}\right)$.

3. Use $b_1 = \dfrac{f_1 c \sqrt{4\kappa + (f_1 c)^2} + (x_2^2 - x_1^2) + (y_2^2 - y_1^2)}{2(y_2 - y_1)}$.

4. Use $b_2 = \dfrac{f_2 c \left((f_1 + f_2)c + \sqrt{4\kappa + (f_1 c)^2}\right) + (x_3^2 - x_1^2) + (y_3^2 - y_1^2)}{2(y_3 - y_1)}$.

5. Solve

$$\kappa = \frac{d_1^2 - (f_1 c)^2}{2} + \left\{ \begin{array}{c} \left(x_1 - \left(\frac{b_2 - b_1}{m_1 - m_2}\right)\right)\left(x_2 - \left(\frac{b_2 - b_1}{m_1 - m_2}\right)\right) \\ + \left(y_1 - \left(m_1\left(\frac{b_2 - b_1}{m_1 - m_2}\right) + b_1\right)\right)\left(y_2 - \left(m_1\left(\frac{b_2 - b_1}{m_1 - m_2}\right) + b_1\right)\right) \end{array} \right\}$$

   for $\kappa$; call this value $\kappa_0$.

6. Evaluate $b_1$ and $b_2$ with $\kappa = \kappa_0$; call these values $\beta_1$ and $\beta_2$, respectively.

7. Then

$$(x_0, y_0) = \left(\frac{\beta_2 - \beta_1}{m_1 - m_2}, m_1\left(\frac{\beta_2 - \beta_1}{m_1 - m_2}\right) + \beta_1\right)$$

## 9. When $y_1 = y_2$, $y_1 \neq y_3$, and $d_i > |f_i|\, c$, for $i = 1, 2$

1. Use $r_1 = \dfrac{\pm f_1 c \sqrt{4\kappa + (f_1 c)^2} + (x_2^2 - x_1^2)}{2(x_2 - x_1)}$.

2. Use $r_2 = \dfrac{1}{2(y_3 - y_1)}\left( \begin{array}{c} f_2 c\left((f_1 + f_2)c \pm \sqrt{4\kappa + (f_1 c)^2}\right) + (x_3^2 - x_1^2) + (y_3^2 - y_1^2) \\ - \left(\pm f_1 c \sqrt{4\kappa + (f_1 c)^2} + (x_2^2 - x_1^2)\right)\left(\frac{x_3 - x_1}{x_2 - x_1}\right) \end{array} \right)$.

3. Solve

$$\kappa = \frac{d_1^2 - (f_1 c)^2}{2} + (x_1 - r_1)(x_2 - r_1) + (y_1 - r_2)(y_2 - r_2)$$

   for $\kappa$; call this value $\kappa_0$.

4. Evaluate $r_1$ and $r_2$ with $\kappa = \kappa_0$; call these values $\gamma_1$ and $\gamma_2$, respectively.

5. Then

$$(x_0, y_0) = (\gamma_1, \gamma_2)$$

**10.**   **When $y_1 \neq y_2$, $y_1 = y_3$, and $d_i > |f_i|\, c$, for $i = 1, 2$**

1. Use $r_1 = \frac{f_2 c^2 (f_1 + f_2) \pm \sqrt{4\kappa_0 + (f_1 c)^2} + (x_3^2 - x_1^2)}{2(x_3 - x_1)}$.

2. Use $r_2 = \frac{1}{2(y_2 - y_1)} \left( \begin{array}{c} \pm f_1 c \sqrt{4\kappa_0 + (f_1 c)^2} + (x_2^2 - x_1^2) + (y_2^2 - y_1^2) \\ - \left( f_2 c \left( (f_1 + f_2) c \pm \sqrt{4\kappa_0 + (f_1 c)^2} \right) + (x_3^2 - x_1^2) \right) \left( \frac{x_2 - x_1}{x_3 - x_1} \right) \end{array} \right)$.

3. Solve

$$\kappa = \frac{d_1^2 - (f_1 c)^2}{2} + (x_1 - r_1)(x_2 - r_1) + (y_1 - r_2)(y_2 - r_2)$$

    for $\kappa$; call this value $\kappa_0$.

4. Evaluate $r_1$ and $r_2$ with $\kappa = \kappa_0$; call these values $\gamma_1$ and $\gamma_2$, respectively.

5. Then

$$(x_0, y_0) = (\gamma_1, \gamma_2)$$

## 11.   Calculated Target Position Examples

Suppose $(x_1, y_1) = (2, 3)$, $(x_2, y_2) = (4, 2)$, and $(x_3, y_3) = (3, 1)$, and observed data $t_1 = 1.6$, $t_2 = 2$, $t_3 = 1.75$. For demonstration purposes only, let $c = 1$ (see Figure ). Then $y_1 \neq y_2$, $y_1 \neq y_3$, and $d_i > |f_i|\, c$, for $i = 1, 2$.

1. We have $f_1 = t_1 - t_2 = 1.6 - 2 = -0.40$, and $f_2 = t_1 - t_3 = 1.6 - 1.75 = -0.15$.

2. Then $d_1 = \sqrt{(2-4)^2 + (3-2)^2} = \sqrt{5}$ and $d_2 = \sqrt{(2-3)^2 + (3-1)^2} = \sqrt{5}$.

3. And $m_1 = -\left( \frac{4-2}{2-3} \right) = 2$.

4. Then $m_2 = -\left( \frac{3-2}{1-3} \right) = \frac{1}{2}$.

5. Also $b_1 = \frac{-0.40\sqrt{4\kappa + (-0.40)^2} + (4^2 - 2^2) + (2^2 - 3^2)}{2(2-3)} = \frac{2}{25}\sqrt{25\kappa + 1} - \frac{7}{2}$. Note that the positive sign has been chosen.

6. And $b_2 = \frac{-0.15\left( (-0.40 - 0.15) + \sqrt{4\kappa + (-0.40)^2} \right) + (3^2 - 2^2) + (1^2 - 3^2)}{2(1-3)} = \frac{1167}{1600} + \frac{3}{200}\sqrt{25\kappa + 1}$. Note that the positive sign has been chosen.

7. Solving for $\kappa_0$ gives us

$$\kappa_0 = \frac{26454174801}{18790926400} + \frac{561}{2348865800}\sqrt{292162046321} = 1.5369135\cdots$$

8. Therefore

$$\beta_1 = \frac{2}{25}\sqrt{\frac{27205811857}{751637056} + \frac{561}{93954632}\sqrt{292162046321}} - \frac{7}{2} = -2.9976991\cdots$$

    and

$$\beta_2 = \frac{1167}{1600} + \frac{3}{200}\sqrt{\frac{27205811857}{751637056} + \frac{561}{93954632}\sqrt{292162046321}} = 0.8235564\cdots$$

9. So

$$(x_0, y_0) = \left( \frac{\beta_2 - \beta_1}{m_1 - m_2}, m_1 \left( \frac{\beta_2 - \beta_1}{m_1 - m_2} \right) + \beta_1 \right) = \boxed{(2.5475037\cdots, 2.0973083\cdots)}$$

    to 7 decimal places (see Figure 8).

Now suppose $(x_1, y_1) = (2, 3)$, $(x_2, y_2) = (4, 3)$, and $(x_3, y_3) = (3, 1)$, and the observed data is $t_1 = 1.6$, $t_2 = 2$, $t_3 = 1.75$. Note that $y_1 = y_2$ and $y_1 \neq y_3$, and $d_i > |f_i|\, c$, for $i = 1, 2$. For demonstration purposes only, let $c = 1$.

1. We have $f_1 = t_1 - t_2 = 1.6 - 2 = -0.40$, and $f_2 = t_1 - t_3 = 1.6 - 1.75 = -0.15$.

2. Then $d_1 = \sqrt{(2-4)^2 + (3-3)^2} = 2$ and $d_2 = \sqrt{(2-3)^2 + (3-1)^2} = \sqrt{5}$.

3. This means $r_1 = \frac{-0.40\sqrt{4\kappa + (-0.40)^2} + (16 - 4)}{2(4-2)} = -\frac{1}{25}\sqrt{25\kappa + 1} + 3$. Note that the positive sign has been chosen.
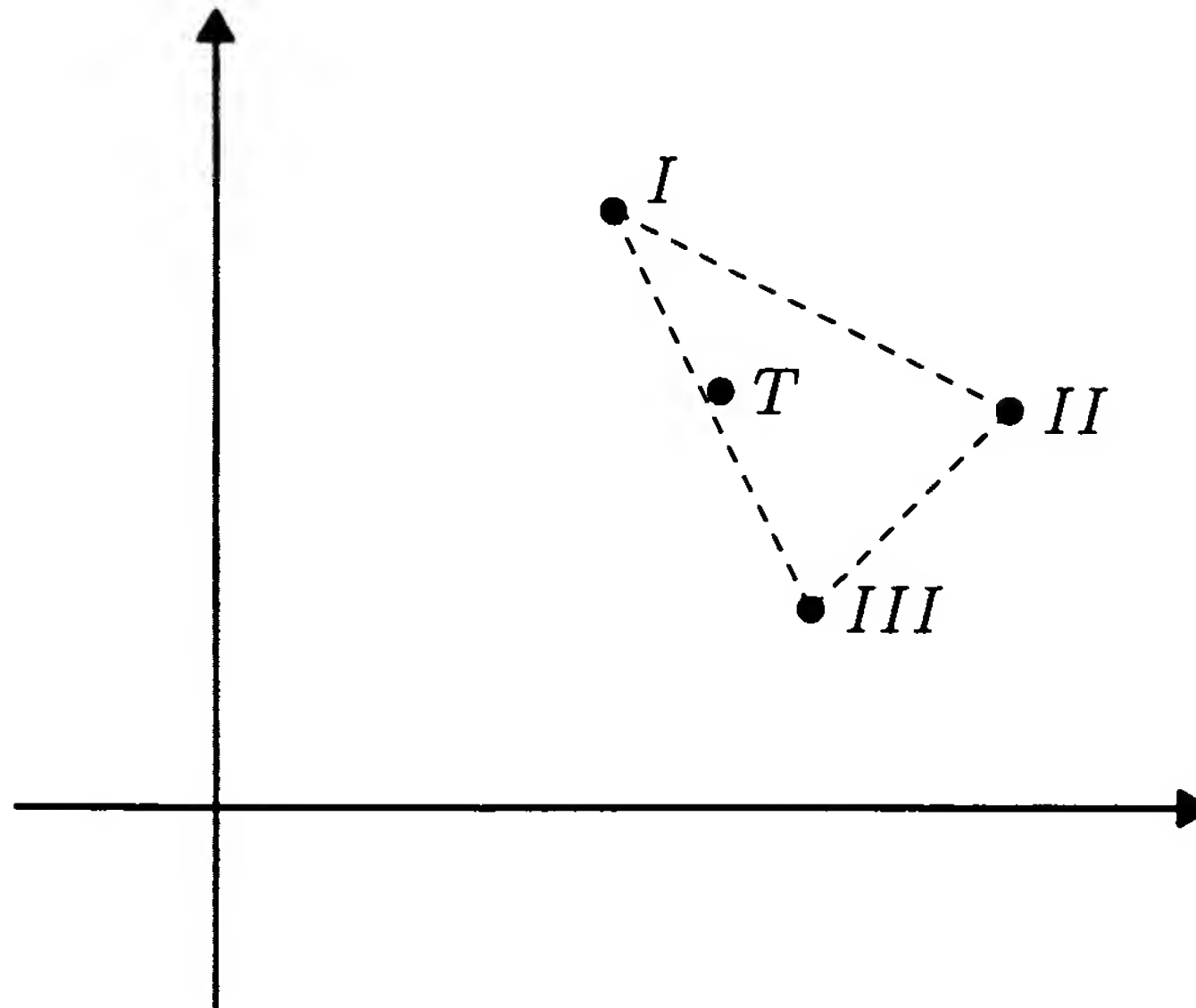
**Figure 8**: Example Target Calculation Result when $y_1 \neq y_2$, $y_1 \neq y_3$, and $d_i > |f_i| c$, for $i = 1, 2$.

4. And

$$r_2 = \frac{1}{2(1-3)} \left( \begin{array}{c} -0.15 \left( (-0.40 - 0.15) \pm \sqrt{4\kappa + (-0.40)^2} \right) + (9 - 4) + (1 - 9) \\ - \left( -0.40\sqrt{4\kappa + (-0.40)^2} + (16 - 4) \right) \left( \frac{3-2}{4-2} \right) \end{array} \right)$$
$$= \frac{1}{4} \left( -0.15 * 0.55 - 0.05\sqrt{4\kappa + 0.16} + 9 \right)$$
$$= \frac{3567}{1600} - \frac{1}{200}\sqrt{25\kappa + 1}$$

Note that the positive sign has been chosen.

5. Solving for $\kappa_0$ gives us

$$\kappa_0 = \frac{5958301793}{3769960000} + \frac{1233\sqrt{23857734}}{117811250}$$

6. Therefore

$$\gamma_1 = -\frac{1}{25}\sqrt{\frac{6109100193}{150798400} + \frac{1233\sqrt{23857734}}{4712450}} + 3 = 2.7414202\cdots$$

and

$$\gamma_2 = \frac{3567}{1600} - \frac{1}{200}\sqrt{\frac{6109100193}{150798400} + \frac{1233\sqrt{23857734}}{4712450}} = 2.1975504\cdots$$

7. So

$$(x_0, y_0) = \boxed{\boxed{(2.7414202\cdots, 2.1975504\cdots)}}$$

to 7 decimal places (see Figure 9).

## 12.   Algorithm Implementation Summary
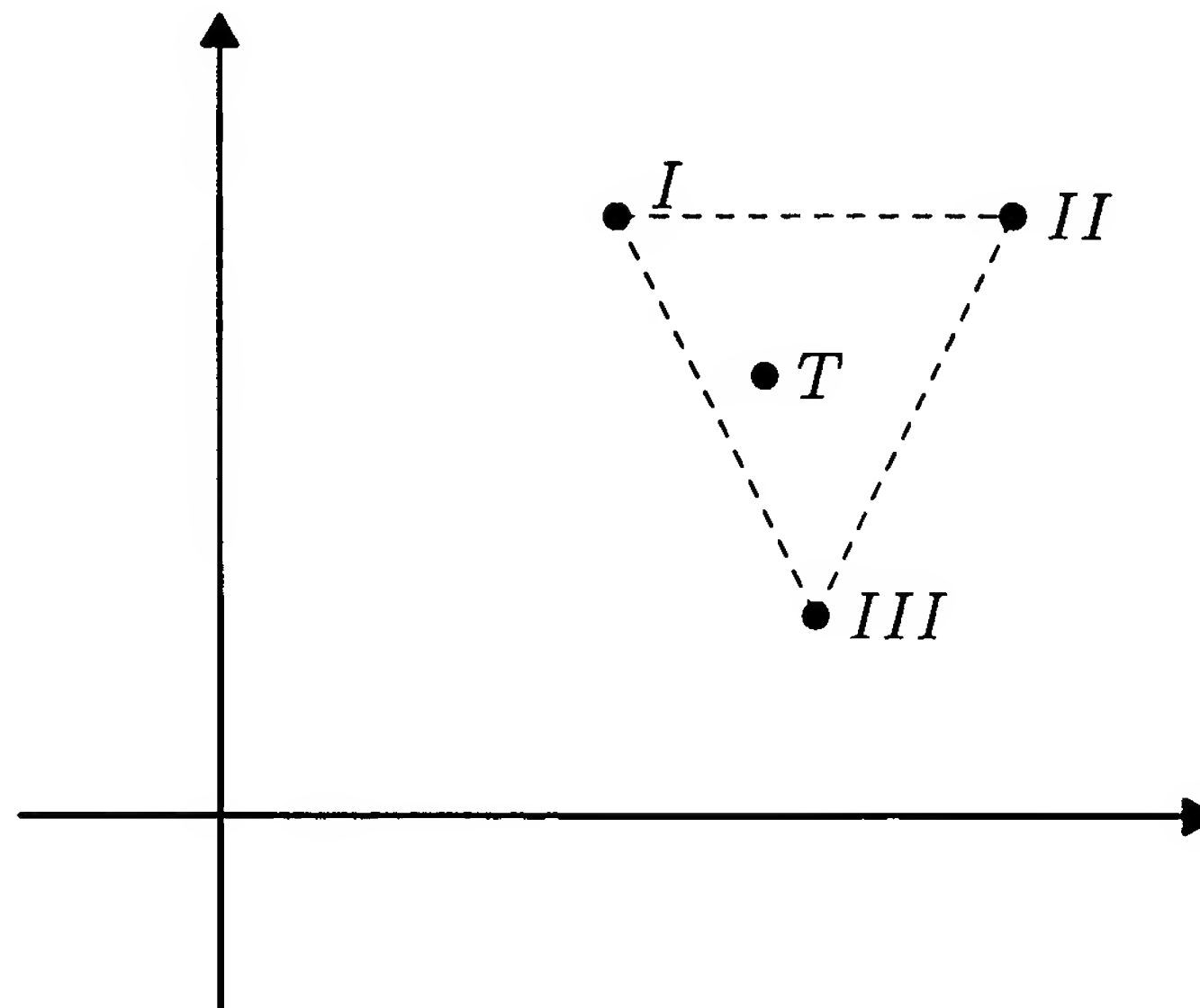
**Figure 9**: Example Target Calculation Result when $y_1 = y_2$, $y_1 \neq y_3$, and $d_i > |f_i| c$, for $i = 1, 2$.

## 13.   Target Position Report Calculation

A Target Position Report is the calculated $(x, y)$ position value based on the arrival times detected at the SDU's, and the position information for the SDU's involved in the calculation. There are no adjustments made for the uncertainty in the arrival time data, as such issues are addressed by the Error Likelihood Ellipse methods.

The principal calculation policy in the PQIC OSMMTS System is to base a Target Position Report on the choice of SDU arrival times that maximizes the likelihood that the calculated Target Position Report is as accurate, i.e., close to the actual target position, as possible. Since the PQIC OSMMTS SDU's do not change position as a function of time when the system is operating, this optimization may be achieved by implementing the PQIC OSMMTS Target Position Report Standard Methodology, which means pre-establishing the calculation algorithm that the system will utilize when a particular combination of SDU arrival times are used to calculate the Target Position Report.

When error conditions are calculated, e.g., in the Error Likelihood Ellipse, the same algorithm set shall be used with appropriately adjusted arrival time data.

## 14.   Target Position Report Standard Methodology

The PQIC OSMMTS Target Position Report Standard Methodology for calculating a Target Position Report begins with the use of a position-specific combination of the SDU's taken three at a time. Based on this position information, the Target Position Report algorithm calculates a $(x, y)$ position value for the given $t_1, t_2, t_3$ values. This algorithm is specific to the combination of SDU's that receive the arrival time data. Therefore, for $n$-many SDU positions, there must be $\binom{n}{3}$-many algorithms pre-established for use by the Target Position Report calculation device. For example, if there are 5 SDU positions, then $\binom{5}{3=10}$ separate algorithms must be established to calculate a Target Position Report regardless of which three SDU's are involved in the calculation.

The set of algorithms generated for a particular set of SDU positions differ only by the number of terms and the coefficients of the terms involved in the sum. An effective organizing system that addresses the numerical issues related to such a complicated sum may be found later in this document.

### 15. Target Position Report MAPLE Algorithm

Given the position information of the SDU's that receive the arrival time data, the following MAPLE algorithm calculates the Target Position Report based on the arrival times. For the purposes of this document, only example algorithms shall be given based on example position information. The completely generic form of the Target Position Report MAPLE Algorithm shall be documented in a separate PQIC OSMMTS technical memorandum.

In general, each term of the Target Position Report calculation has the form

$$(\text{sign}) \times (\text{coefficient}) \times (\text{ratio}) \times c^{p_0} \times t_1^{p_1} \times t_2^{p_2} \times t_3^{p_3}$$

for $p_i = 0, 1, 2, ...,$ where $p_0 = p_1 + p_2 + p_3$. The sign is either positive or negative 1, the coefficient is a positive integer, and the ratio is either 1 or a ration of previously calculated subexpressions. If the coefficient is allowed to be nonnegative, i.e., take on the value 0, then, in general, each Target Position Report may be expressed as the sum of the same number of such terms, where the number of such terms depends on the SDU position information.

The Target Position Report terms may be organized into convenient subassemblies for calculation efficiency by using the tables found below as worksheets, as demonstrated later in this document.

### 16. ⟨TPR Calculation 16⟩ ≡

```
osmmts: = proc(n1, n2, n3)
local f1, f2, d1, m1, m2, b1, b2, k, k0x, beta1, beta2;
global c, x0, y0, x1, y1, x2, y2, x3, y3, s1, s2, s3, k0, cth0, th0;
options 'Copyright 2003 PQI Consulting All Rights Reserved';
description "OSMMTS␣Target␣Position␣Report␣and␣Supporting␣Information";
f1: = n1 − n2; f2: = n1 − n3;
d1: = sqrt((x2 − x1) ⊕ 2 + (y2 − y1) ⊕ 2);
m1: = −((x2 − x1)/(y2 − y1));
m2: = −((x3 − x1)/(y3 − y1));
b1: = (f1 * c * sqrt(4 * k + (f1 * c) ⊕ 2) +(x2 ⊕ 2 − x1 ⊕ 2) + (y2 ⊕ 2 − y1 ⊕ 2))/(2 * (y2 − y1));
b2: = (f2 * c * ((f1 + f2) * c + sqrt(4 * k + (f1 * c) ⊕ 2)) +(x3 ⊕ 2 − x1 ⊕ 2) + (y3 ⊕ 2 − y1 ⊕ 2))/(2 * (y3 − y1));
k0x: = °
{solve(k = ((d1 ⊕ 2 − (f1 * c) ⊕ 2)/2) +(x1 − ((b2 − b1)/(m1 − m2))) *(x2 − ((b2 − b1)/(m1 − m2)))
      +(y1 − (m1 * ((b2 − b1)/(m1 − m2)) + b1)) *(y2 − (m1 * ((b2 − b1)/(m1 − m2)) + b1)), k)° };
if (nops(k0x) = 0) then
k0: = undefined;
elif (nops(k0x) > 1) then
if (k0x[2] = conjugate(k0x[1])) then
k0: = undefined;
else
k0: = min(op(1, k0x), op(2, k0x));
end if ;
else
k0: = k0x[1];
end if ;
beta1: = eval(b1, k = k0);
beta2: = eval(b2, k = k0);
x0: = (beta2 − beta1)/(m1 − m2);
y0: = m1 * x0 + beta1;
cth0: = 1 − ((d1 ⊕ 2 − (f1 * c) ⊕ 2)/(2 * k0));
th0: = arccos(cth0);
s1: = sqrt((x1 − x0) ⊕ 2 + (y1 − y0) ⊕ 2);
s2: = sqrt((x2 − x0) ⊕ 2 + (y2 − y0) ⊕ 2);
s3: = sqrt((x3 − x0) ⊕ 2 + (y3 − y0) ⊕ 2);
return x0, y0, k0, k0x, nops(k0x), cth0, th0, s1, s2, s3;
```

*endproc*;

This code is used in chunk 12.

## 17.    Target Position Report Example #1

**18.**   $\langle$ TPR Example #1 SDU Coordinates 18 $\rangle \equiv$
$(x1, y1)$: $= (2,3)$; $(x2, y2)$: $= (4,2)$; $(x3, y3)$: $= (3,1)$;

This code is used in chunk 12.

## 19.    Target Position Report MAPLE Calculation For Example #1

```
>  c:=1; (x1,y1):=(2,3);(x2,y2):=(4,2);(x3,y3):=(3,1);
```

$$c := 1$$
$$x1,\, y1 := 2,\, 3$$
$$x2,\, y2 := 4,\, 2$$
$$x3,\, y3 := 3,\, 1$$

```
>  osmmts(t1,t2,t3)[1];
```

## 20.    *X*-Value Of Target Position Report For Example #1

$$-\frac{1}{6}(t1 - t3)\,c((2\,t1 - t2 - t3)\,c + ((-(50 - \frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - 26\,c^4\,t1^3\,t2 - 30\,t1\,c^4\,t3^3$$

$$+ 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3 - 12\,c^4\,t2^3\,t1 + 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2$$

$$+ 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3 - \frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1}$$

$$- \frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1} - 62\,c^4\,t1^3\,t3 - \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1} + 10\,t3^3\,c^4\,t2$$

$$+ \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1} + 5\,c^4\,t3^4 - \frac{5\,\%4\,c\,t3}{\%1} + \frac{\%4\,c\,t1}{\%1} - \frac{5\,\%4\,c^3\,t3^3}{\%1} + \frac{4\,\%4\,c\,t2}{\%1}$$

$$+ \frac{2\,\%4\,c^3\,t1^3}{\%1} - 18\,c^2\,t2^2 + 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3$$

$$- 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/(\%1) - (50 - 26\,c^4\,t1^3\,t2$$

$$- 30\,t1\,c^4\,t3^3 + 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3 - 12\,c^4\,t2^3\,t1 + 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2$$

$$+ 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3 - 62\,c^4\,t1^3\,t3 + 10\,t3^3\,c^4\,t2$$

$$- \frac{4\,\%3\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%3\,c^3\,t1^2\,t2}{\%1} - \frac{13\,\%3\,c^3\,t1^2\,t3}{\%1} - \frac{6\,\%3\,c^3\,t1\,t2\,t3}{\%1}$$

$$+ \frac{16\,\%3\,t1\,c^3\,t3^2}{\%1} + \frac{4\,\%3\,t3\,c^3\,t2^2}{\%1} - \frac{\%3\,t3^2\,c^3\,t2}{\%1} + 5\,c^4\,t3^4 + \frac{\%3\,c\,t1}{\%1}$$

$$- \frac{5\,\%3\,c^3\,t3^3}{\%1} + \frac{4\,\%3\,c\,t2}{\%1} + \frac{2\,\%3\,c^3\,t1^3}{\%1} - \frac{5\,\%3\,c\,t3}{\%1} - 18\,c^2\,t2^2 + 2\,c^2\,t1^2$$

$$+ 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/($$

$$\%1)) + c^2\,t1^2 - 2\,c^2\,t1\,t2 + c^2\,t2^2)^{(1/2)}) + \frac{17}{6} + \frac{1}{3}(t1 - t2)\,c((-(50$$

$$- \frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - 26\,c^4\,t1^3\,t2 - 30\,t1\,c^4\,t3^3 + 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3$$

$$- 12\,c^4\,t2^3\,t1 + 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2 + 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3$$

$$- \frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1} - \frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1} - 62\,c^4\,t1^3\,t3$$

$$- \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1} + 10\,t3^3\,c^4\,t2 + \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1} + 5\,c^4\,t3^4 - \frac{5\,\%4\,c\,t3}{\%1}$$

$$+ \frac{\%4\,c\,t1}{\%1} - \frac{5\,\%4\,c^3\,t3^3}{\%1} + \frac{4\,\%4\,c\,t2}{\%1} + \frac{2\,\%4\,c^3\,t1^3}{\%1} - 18\,c^2\,t2^2 + 2\,c^2\,t1^2$$

$$+ 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/($$

$$\%1) - (50 - 26\,c^4\,t1^3\,t2 - 30\,t1\,c^4\,t3^3 + 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3 - 12\,c^4\,t2^3\,t1$$

$$+ 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2 + 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3 - 62\,c^4\,t1^3\,t3$$

$$+ 10\,t3^3\,c^4\,t2 - \frac{4\,\%3\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%3\,c^3\,t1^2\,t2}{\%1} - \frac{13\,\%3\,c^3\,t1^2\,t3}{\%1}$$

$$- \frac{6\,\%3\,c^3\,t1\,t2\,t3}{\%1} + \frac{16\,\%3\,t1\,c^3\,t3^2}{\%1} + \frac{4\,\%3\,t3\,c^3\,t2^2}{\%1} - \frac{\%3\,t3^2\,c^3\,t2}{\%1} + 5\,c^4\,t3^4$$

$$+ \frac{\%3\,c\,t1}{\%1} - \frac{5\,\%3\,c^3\,t3^3}{\%1} + \frac{4\,\%3\,c\,t2}{\%1} + \frac{2\,\%3\,c^3\,t1^3}{\%1} - \frac{5\,\%3\,c\,t3}{\%1} - 18\,c^2\,t2^2$$

$$+ 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4$$

$$+ 5\,c^4\,t2^4)/(\%1)) + c^2\,t1^2 - 2\,c^2\,t1\,t2 + c^2\,t2^2)^{(1/2)}$$

$$\%1 := 5\,c^2\,t2^2 + 2\,c^2\,t1^2 + 5\,c^2\,t3^2 - 9 - 2\,t1\,c^2\,t3 - 2\,c^2\,t1\,t2 - 8\,t3\,c^2\,t2$$

$$\%2 := 50 - c^6\,t1^2\,t2^4 + 12\,c^6\,t1^5\,t2 + 6\,c^6\,t1^3\,t2^3 - 13\,c^6\,t1^4\,t2^2 + 12\,c^6\,t1^5\,t3 - c^6\,t1^2\,t3^4$$

$$+ 6\,c^6\,t1^3\,t3^3 - 13\,c^6\,t1^4\,t3^2 - 16\,c^4\,t1^3\,t2 - 14\,c^6\,t1^2\,t3^3\,t2 + 34\,c^6\,t1^3\,t3\,t2^2$$

$$- 34\,c^6\,t1^4\,t3\,t2 - t3^2\,c^6\,t2^4 - 2\,t3^3\,c^6\,t2^3 - t3^4\,c^6\,t2^2 + 34\,c^6\,t1^3\,t3^2\,t2$$

$$- 30\,t1\,c^4\,t3^3 + 8\,c^4\,t1^2\,t2^2 - 30\,c^6\,t1^2\,t3^2\,t2^2 - 20\,t1\,c^2\,t3 - 10\,c^4\,t2^3\,t1$$

$$- 14\,c^6\,t1^2\,t2^3\,t3 + 20\,c^2\,t1\,t2 + 10\,c^6\,t1\,t2^2\,t3^3 + 2\,c^6\,t1\,t2^4\,t3 + 10\,c^6\,t1\,t2^3\,t3^2$$

$$+ 68\,c^4\,t1^2\,t3^2 + 2\,t1\,c^6\,t3^4\,t2 + 8\,t3^2\,c^4\,t2^2 + 50\,t3\,c^2\,t2 - 10\,t3\,c^4\,t2^3 - 4\,c^6\,t1^6$$

$$- 56\,c^4\,t1^3\,t3 + 10\,t3^3\,c^4\,t2 + 5\,c^4\,t3^4 - 35\,c^2\,t2^2 + 32\,c^4\,t1^2\,t3\,t2 + 14\,c^4\,t1\,t2^2\,t3$$

$$- 46\,c^4\,t1\,t2\,t3^2 - 15\,c^2\,t3^2 + 18\,c^4\,t1^4 + 5\,c^4\,t2^4$$

$$\%3 := 26\,c^3\,t1^2\,t3 - 14\,c^3\,t1^2\,t2 + 8\,c^3\,t1\,t2^2 - 32\,t1\,c^3\,t3^2 + 2\,t3^2\,c^3\,t2 - 8\,t3\,c^3\,t2^2$$

$$- 8\,c\,t2 + 10\,c^3\,t3^3 - 2\,c\,t1 + 10\,c\,t3 + 12\,c^3\,t1\,t2\,t3 - 4\,c^3\,t1^3 - 6\,\sqrt{\%2}$$

$$\%4 := 26\,c^3\,t1^2\,t3 - 14\,c^3\,t1^2\,t2 + 8\,c^3\,t1\,t2^2 - 32\,t1\,c^3\,t3^2 + 2\,t3^2\,c^3\,t2 - 8\,t3\,c^3\,t2^2$$

$$- 8\,c\,t2 + 10\,c^3\,t3^3 - 2\,c\,t1 + 10\,c\,t3 + 12\,c^3\,t1\,t2\,t3 - 4\,c^3\,t1^3 + 6\,\sqrt{\%2}$$

## 21.  $Y$-Value Of Target Position Report For Example #1

```
> osmmts(t1,t2,t3)[2];
```

$$-\frac{1}{3}(t1 - t3)\,c\,((2\,t1 - t2 - t3)\,c + ((-(50 - \frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - 26\,c^4\,t1^3\,t2 - 30\,t1\,c^4\,t3^3$$

$$+ 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3 - 12\,c^4\,t2^3\,t1 + 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2$$

$$+ 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3 - \frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1}$$

$$- \frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1} - 62\,c^4\,t1^3\,t3 - \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1} + 10\,t3^3\,c^4\,t2$$

$$+ \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1} + 5\,c^4\,t3^4 - \frac{5\,\%4\,c\,t3}{\%1} + \frac{\%4\,c\,t1}{\%1} - \frac{5\,\%4\,c^3\,t3^3}{\%1} + \frac{4\,\%4\,c\,t2}{\%1}$$

$$+ \frac{2\,\%4\,c^3\,t1^3}{\%1} - 18\,c^2\,t2^2 + 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3$$

$$- 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/(\%1), -(50 - 26\,c^4\,t1^3\,t2$$

$$- 30\,t1\,c^4\,t3^3 + 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3 - 12\,c^4\,t2^3\,t1 + 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2$$

$$+ 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3 - 62\,c^4\,t1^3\,t3 + 10\,t3^3\,c^4\,t2$$

$$- \frac{4\,\%3\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%3\,c^3\,t1^2\,t2}{\%1} - \frac{13\,\%3\,c^3\,t1^2\,t3}{\%1} - \frac{6\,\%3\,c^3\,t1\,t2\,t3}{\%1}$$

$$+ \frac{16\,\%3\,t1\,c^3\,t3^2}{\%1} + \frac{4\,\%3\,t3\,c^3\,t2^2}{\%1} - \frac{\%3\,t3^2\,c^3\,t2}{\%1} + 5\,c^4\,t3^4 + \frac{\%3\,c\,t1}{\%1}$$

$$- \frac{5\,\%3\,c^3\,t3^3}{\%1} + \frac{4\,\%3\,c\,t2}{\%1} + \frac{2\,\%3\,c^3\,t1^3}{\%1} - \frac{5\,\%3\,c\,t3}{\%1} - 18\,c^2\,t2^2 + 2\,c^2\,t1^2$$

$$+ 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/($$

$$\%1)) + c^2\,t1^2 - 2\,c^2\,t1\,t2 + c^2\,t2^2)^{(1/2)}) + \frac{13}{6} + \frac{1}{6}(t1 - t2)\,c((-(50$$

$$- \frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - 26\,c^4\,t1^3\,t2 - 30\,t1\,c^4\,t3^3 + 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3$$

$$- 12\,c^4\,t2^3\,t1 + 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2 + 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3$$

$$- \frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1} - \frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1} - 62\,c^4\,t1^3\,t3$$

$$- \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1} + 10\,t3^3\,c^4\,t2 + \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1} + 5\,c^4\,t3^4 - \frac{5\,\%4\,c\,t3}{\%1}$$

$$+ \frac{\%4\,c\,t1}{\%1} - \frac{5\,\%4\,c^3\,t3^3}{\%1} + \frac{4\,\%4\,c\,t2}{\%1} + \frac{2\,\%4\,c^3\,t1^3}{\%1} - 18\,c^2\,t2^2 + 2\,c^2\,t1^2$$

$$+ 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/($$

$$\%1), -(50 - 26\,c^4\,t1^3\,t2 - 30\,t1\,c^4\,t3^3 + 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3 - 12\,c^4\,t2^3\,t1$$

$$+ 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2 + 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3 - 62\,c^4\,t1^3\,t3$$

$$+ 10\,t3^3\,c^4\,t2 - \frac{4\,\%3\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%3\,c^3\,t1^2\,t2}{\%1} - \frac{13\,\%3\,c^3\,t1^2\,t3}{\%1}$$

$$- \frac{6\,\%3\,c^3\,t1\,t2\,t3}{\%1} + \frac{16\,\%3\,t1\,c^3\,t3^2}{\%1} + \frac{4\,\%3\,t3\,c^3\,t2^2}{\%1} - \frac{\%3\,t3^2\,c^3\,t2}{\%1} + 5\,c^4\,t3^4$$

$$+ \frac{\%3\,c\,t1}{\%1} - \frac{5\,\%3\,c^3\,t3^3}{\%1} + \frac{4\,\%3\,c\,t2}{\%1} + \frac{2\,\%3\,c^3\,t1^3}{\%1} - \frac{5\,\%3\,c\,t3}{\%1} - 18\,c^2\,t2^2$$

$$+ 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4$$

$$+ 5\,c^4\,t2^4)/(\%1)) + c^2\,t1^2 - 2\,c^2\,t1\,t2 + c^2\,t2^2)^{(1/2)}$$

$$\%1 := 5\,c^2\,t2^2 + 2\,c^2\,t1^2 + 5\,c^2\,t3^2 - 9 - 2\,t1\,c^2\,t3 - 2\,c^2\,t1\,t2 - 8\,t3\,c^2\,t2$$

$$\%2 := 50 - c^6\,t1^2\,t2^4 + 12\,c^6\,t1^5\,t2 + 6\,c^6\,t1^3\,t2^3 - 13\,c^6\,t1^4\,t2^2 + 12\,c^6\,t1^5\,t3 - c^6\,t1^2\,t3^4$$

$$+ 6\,c^6\,t1^3\,t3^3 - 13\,c^6\,t1^4\,t3^2 - 16\,c^4\,t1^3\,t2 - 14\,c^6\,t1^2\,t3^3\,t2 + 34\,c^6\,t1^3\,t3\,t2^2$$

$$- 34\,c^6\,t1^4\,t3\,t2 - t3^2\,c^6\,t2^4 - 2\,t3^3\,c^6\,t2^3 - t3^4\,c^6\,t2^2 + 34\,c^6\,t1^3\,t3^2\,t2$$

$$- 30\,t1\,c^4\,t3^3 + 8\,c^4\,t1^2\,t2^2 - 30\,c^6\,t1^2\,t3^2\,t2^2 - 20\,t1\,c^2\,t3 - 10\,c^4\,t2^3\,t1$$

$$- 14\,c^6\,t1^2\,t2^3\,t3 + 20\,c^2\,t1\,t2 + 10\,c^6\,t1\,t2^2\,t3^3 + 2\,c^6\,t1\,t2^4\,t3 + 10\,c^6\,t1\,t2^3\,t3^2$$

$$+ 68\,c^4\,t1^2\,t3^2 + 2\,t1\,c^6\,t3^4\,t2 + 8\,t3^2\,c^4\,t2^2 + 50\,t3\,c^2\,t2 - 10\,t3\,c^4\,t2^3 - 4\,c^6\,t1^6$$

$$- 56\,c^4\,t1^3\,t3 + 10\,t3^3\,c^4\,t2 + 5\,c^4\,t3^4 - 35\,c^2\,t2^2 + 32\,c^4\,t1^2\,t3\,t2 + 14\,c^4\,t1\,t2^2\,t3$$

$$- 46\,c^4\,t1\,t2\,t3^2 - 15\,c^2\,t3^2 + 18\,c^4\,t1^4 + 5\,c^4\,t2^4$$

$$\%3 := 26\,c^3\,t1^2\,t3 - 14\,c^3\,t1^2\,t2 + 8\,c^3\,t1\,t2^2 - 32\,t1\,c^3\,t3^2 + 2\,t3^2\,c^3\,t2 - 8\,t3\,c^3\,t2^2$$
$$- 8\,c\,t2 + 10\,c^3\,t3^3 - 2\,c\,t1 + 10\,c\,t3 + 12\,c^3\,t1\,t2\,t3 - 4\,c^3\,t1^3 - 6\,\sqrt{\%2}$$
$$\%4 := 26\,c^3\,t1^2\,t3 - 14\,c^3\,t1^2\,t2 + 8\,c^3\,t1\,t2^2 - 32\,t1\,c^3\,t3^2 + 2\,t3^2\,c^3\,t2 - 8\,t3\,c^3\,t2^2$$
$$- 8\,c\,t2 + 10\,c^3\,t3^3 - 2\,c\,t1 + 10\,c\,t3 + 12\,c^3\,t1\,t2\,t3 - 4\,c^3\,t1^3 + 6\,\sqrt{\%2}$$

## 22. Target Position Report Example #2

## 23. ⟨TPR Example #2 SDU Coordinates 23⟩ ≡
$(x1, y1)\colon = (3, 2);\ (x2, y2)\colon = (2, 4);\ (x3, y3)\colon = (1, 3);$
This code is used in chunk 12.

## 24. Target Position Report MAPLE Calculation For Example #2

```
>   c:=1; (x1,y1):=(3,2);(x2,y2):=(2,4);(x3,y3):=(1,3);
```

$$c := 1$$
$$x1,\ y1 := 3,\ 2$$
$$x2,\ y2 := 2,\ 4$$
$$x3,\ y3 := 1,\ 3$$

```
>   osmmts(t1,t2,t3)[1];
```

## 25. X-Value Of Target Position Report For Example #2

$$-\frac{1}{3}(t1 - t3)\,c\Big((2\,t1 - t2 - t3)\,c + \big(4\min\big(-(50 + 26\,c^2\,t1\,t2 - 8\,t3\,c^4\,t2^3 - 30\,t1\,c^2\,t3$$

$$+ 10\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2 - \frac{6\,\%3\,c^3\,t1\,t2\,t3}{\%1} + \frac{4\,\%3\,t3\,c^3\,t2^2}{\%1} - \frac{\%3\,t3^2\,c^3\,t2}{\%1}$$

$$+ \frac{7\,\%3\,c^3\,t1^2\,t2}{\%1} - \frac{4\,\%3\,c^3\,t1\,t2^2}{\%1} + \frac{16\,\%3\,t1\,c^3\,t3^2}{\%1} - \frac{13\,\%3\,c^3\,t1^2\,t3}{\%1}$$

$$- 62\,c^4\,t1^3\,t3 + \frac{\%3\,c\,t1}{\%1} + \frac{2\,\%3\,c^3\,t1^3}{\%1} - \frac{5\,\%3\,c^3\,t3^3}{\%1} - 26\,c^4\,t1^3\,t2 + \frac{4\,\%3\,c\,t2}{\%1}$$

$$- \frac{5\,\%3\,c\,t3}{\%1} + 10\,t3^2\,c^4\,t2^2 - 30\,t1\,c^4\,t3^3 + 70\,c^4\,t1^2\,t3^2 - 12\,c^4\,t2^3\,t1$$

$$+ 16\,c^4\,t1^2\,t2^2 + 5\,c^4\,t3^4 - 18\,c^2\,t2^2 + 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3$$

$$- 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4\big)/(4\,\%1), -(50 + 26\,c^2\,t1\,t2$$

$$- 8\,t3\,c^4\,t2^3 - 30\,t1\,c^2\,t3 + 10\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2 + \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1}$$

$$- \frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1} - \frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1}$$

$$- \frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1} - 62\,c^4\,t1^3\,t3 - 26\,c^4\,t1^3\,t2 + 10\,t3^2\,c^4\,t2^2$$

$$- \frac{5\,\%4\,c^3\,t3^3}{\%1} + \frac{4\,\%4\,c\,t2}{\%1} - \frac{5\,\%4\,c\,t3}{\%1} + \frac{\%4\,c\,t1}{\%1} + \frac{2\,\%4\,c^3\,t1^3}{\%1} - 30\,t1\,c^4\,t3^3$$

$$+ 70\,c^4\,t1^2\,t3^2 - 12\,c^4\,t2^3\,t1 + 16\,c^4\,t1^2\,t2^2 + 5\,c^4\,t3^4 - 18\,c^2\,t2^2 + 2\,c^2\,t1^2$$

$$+ 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4\big)/($$

$$4\,\%1\big)\big) + c^2\,t1^2 - 2\,c^2\,t1\,t2 + c^2\,t2^2\big)^{(1/2)}\Big) + \frac{13}{6} + \frac{1}{6}(t1 - t2)\,c\big(4\min\big(-(50$$

$$+ 26\,c^2\,t1\,t2 - 8\,t3\,c^4\,t2^3 - 30\,t1\,c^2\,t3 + 10\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2$$

$$- \frac{6\,\%3\,c^3\,t1\,t2\,t3}{\%1} + \frac{4\,\%3\,t3\,c^3\,t2^2}{\%1} - \frac{\%3\,t3^2\,c^3\,t2}{\%1} + \frac{7\,\%3\,c^3\,t1^2\,t2}{\%1}$$

$$- \frac{4\,\%3\,c^3\,t1\,t2^2}{\%1} + \frac{16\,\%3\,t1\,c^3\,t3^2}{\%1} - \frac{13\,\%3\,c^3\,t1^2\,t3}{\%1} - 62\,c^4\,t1^3\,t3 + \frac{\%3\,c\,t1}{\%1}$$

$$+ \frac{2\,\%3\,c^3\,t1^3}{\%1} - \frac{5\,\%3\,c^3\,t3^3}{\%1} - 26\,c^4\,t1^3\,t2 + \frac{4\,\%3\,c\,t2}{\%1} - \frac{5\,\%3\,c\,t3}{\%1}$$

$$+ 10\,t3^2\,c^4\,t2^2 - 30\,t1\,c^4\,t3^3 + 70\,c^4\,t1^2\,t3^2 - 12\,c^4\,t2^3\,t1 + 16\,c^4\,t1^2\,t2^2 + 5\,c^4\,t3^4$$

$$- 18\,c^2\,t2^2 + 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2$$

$$+ 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/(4\,\%1),\ -(50 + 26\,c^2\,t1\,t2 - 8\,t3\,c^4\,t2^3 - 30\,t1\,c^2\,t3$$

$$+ 10\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2 + \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1} - \frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1}$$

$$- \frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1} - \frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1}$$

$$- 62\,c^4\,t1^3\,t3 - 26\,c^4\,t1^3\,t2 + 10\,t3^2\,c^4\,t2^2 - \frac{5\,\%4\,c^3\,t3^3}{\%1} + \frac{4\,\%4\,c\,t2}{\%1} - \frac{5\,\%4\,c\,t3}{\%1}$$

$$+ \frac{\%4\,c\,t1}{\%1} + \frac{2\,\%4\,c^3\,t1^3}{\%1} - 30\,t1\,c^4\,t3^3 + 70\,c^4\,t1^2\,t3^2 - 12\,c^4\,t2^3\,t1 + 16\,c^4\,t1^2\,t2^2$$

$$+ 5\,c^4\,t3^4 - 18\,c^2\,t2^2 + 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2$$

$$+ 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/(4\,\%1)) + c^2\,t1^2 - 2\,c^2\,t1\,t2 + c^2\,t2^2)^{(1/2)}$$

$$\%1 := 2\,c^2\,t1^2 + 5\,c^2\,t3^2 + 5\,c^2\,t2^2 - 9 - 8\,t3\,c^2\,t2 - 2\,t1\,c^2\,t3 - 2\,c^2\,t1\,t2$$

$$\%2 := 50 + 20\,c^2\,t1\,t2 - 10\,t3\,c^4\,t2^3 - 20\,t1\,c^2\,t3 + 50\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2 - 56\,c^4\,t1^3\,t3$$

$$- 16\,c^4\,t1^3\,t2 + 8\,t3^2\,c^4\,t2^2 - 30\,t1\,c^4\,t3^3 + 68\,c^4\,t1^2\,t3^2 - 10\,c^4\,t2^3\,t1 + 8\,c^4\,t1^2\,t2^2$$

$$+ 5\,c^4\,t3^4 - 35\,c^2\,t2^2 + 32\,c^4\,t1^2\,t3\,t2 + 14\,c^4\,t1\,t2^2\,t3 - 46\,c^4\,t1\,t2\,t3^2 - 15\,c^2\,t3^2$$

$$+ 18\,c^4\,t1^4 + 5\,c^4\,t2^4 - 4\,c^6\,t1^6 - 34\,c^6\,t1^4\,t3\,t2 + 34\,c^6\,t1^3\,t3\,t2^2 - 14\,c^6\,t1^2\,t3^3\,t2$$

$$- 30\,c^6\,t1^2\,t3^2\,t2^2 + 34\,c^6\,t1^3\,t3^2\,t2 - 14\,c^6\,t1^2\,t2^3\,t3 + 6\,c^6\,t1^3\,t3^3 + 12\,c^6\,t1^5\,t3$$

$$- c^6\,t1^2\,t3^4 - 13\,c^6\,t1^4\,t2^2 + 6\,c^6\,t1^3\,t2^3 + 12\,c^6\,t1^5\,t2 - c^6\,t1^2\,t2^4 - t3^4\,c^6\,t2^2$$

$$- 2\,t3^3\,c^6\,t2^3 - t3^2\,c^6\,t2^4 - 13\,c^6\,t1^4\,t3^2 + 10\,c^6\,t1\,t2^3\,t3^2 + 2\,c^6\,t1\,t2^4\,t3$$

$$+ 10\,c^6\,t1\,t2^2\,t3^3 + 2\,t1\,c^6\,t3^4\,t2$$

$$\%3 := 26\,c^3\,t1^2\,t3 - 14\,c^3\,t1^2\,t2 + 8\,c^3\,t1\,t2^2 - 32\,t1\,c^3\,t3^2 + 2\,t3^2\,c^3\,t2 - 8\,t3\,c^3\,t2^2$$

$$- 4\,c^3\,t1^3 + 10\,c^3\,t3^3 + 10\,c\,t3 - 8\,c\,t2 - 2\,c\,t1 + 12\,c^3\,t1\,t2\,t3 + 6\,\sqrt{\%2}$$

$$\%4 := 26\,c^3\,t1^2\,t3 - 14\,c^3\,t1^2\,t2 + 8\,c^3\,t1\,t2^2 - 32\,t1\,c^3\,t3^2 + 2\,t3^2\,c^3\,t2 - 8\,t3\,c^3\,t2^2$$

$$- 4\,c^3\,t1^3 + 10\,c^3\,t3^3 + 10\,c\,t3 - 8\,c\,t2 - 2\,c\,t1 + 12\,c^3\,t1\,t2\,t3 - 6\,\sqrt{\%2}$$

## 26. $Y$-Value Of Target Position Report For Example #2

```
> osmmts(t1,t2,t3)[2];
```

$$-\frac{1}{6}(t1 - t3)\,c((2\,t1 - t2 - t3)\,c + (4\min(-(50 + 26\,c^2\,t1\,t2 - 8\,t3\,c^4\,t2^3 - 30\,t1\,c^2\,t3$$

$$+ 10\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2 - \frac{6\,\%3\,c^3\,t1\,t2\,t3}{\%1} + \frac{4\,\%3\,t3\,c^3\,t2^2}{\%1} - \frac{\%3\,t3^2\,c^3\,t2}{\%1}$$

$$+ \frac{7\,\%3\,c^3\,t1^2\,t2}{\%1} - \frac{4\,\%3\,c^3\,t1\,t2^2}{\%1} + \frac{16\,\%3\,t1\,c^3\,t3^2}{\%1} - \frac{13\,\%3\,c^3\,t1^2\,t3}{\%1}$$

$$- 62\,c^4\,t1^3\,t3 + \frac{\%3\,c\,t1}{\%1} + \frac{2\,\%3\,c^3\,t1^3}{\%1} - \frac{5\,\%3\,c^3\,t3^3}{\%1} - 26\,c^4\,t1^3\,t2 + \frac{4\,\%3\,c\,t2}{\%1}$$

$$-\frac{5\,\%3\,c\,t3}{\%1} + 10\,t3^2\,c^4\,t2^2 - 30\,t1\,c^4\,t3^3 + 70\,c^4\,t1^2\,t3^2 - 12\,c^4\,t2^3\,t1$$

$$+16\,c^4\,t1^2\,t2^2 + 5\,c^4\,t3^4 - 18\,c^2\,t2^2 + 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3$$

$$-50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/(4\,\%1), -(50 + 26\,c^2\,t1\,t2$$

$$-8\,t3\,c^4\,t2^3 - 30\,t1\,c^2\,t3 + 10\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2 + \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1}$$

$$-\frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1} - \frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1}$$

$$-\frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1} - 62\,c^4\,t1^3\,t3 - 26\,c^4\,t1^3\,t2 + 10\,t3^2\,c^4\,t2^2$$

$$-\frac{5\,\%4\,c^3\,t3^3}{\%1} + \frac{4\,\%4\,c\,t2}{\%1} - \frac{5\,\%4\,c\,t3}{\%1} + \frac{\%4\,c\,t1}{\%1} + \frac{2\,\%4\,c^3\,t1^3}{\%1} - 30\,t1\,c^4\,t3^3$$

$$+70\,c^4\,t1^2\,t3^2 - 12\,c^4\,t2^3\,t1 + 16\,c^4\,t1^2\,t2^2 + 5\,c^4\,t3^4 - 18\,c^2\,t2^2 + 2\,c^2\,t1^2$$

$$+46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/($$

$$4\,\%1)) + c^2\,t1^2 - 2\,c^2\,t1\,t2 + c^2\,t2^2)^{(1/2)}) + \frac{17}{6} + \frac{1}{3}(t1 - t2)\,c(4\min(-(50$$

$$+26\,c^2\,t1\,t2 - 8\,t3\,c^4\,t2^3 - 30\,t1\,c^2\,t3 + 10\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2$$

$$-\frac{6\,\%3\,c^3\,t1\,t2\,t3}{\%1} + \frac{4\,\%3\,t3\,c^3\,t2^2}{\%1} - \frac{\%3\,t3^2\,c^3\,t2}{\%1} + \frac{7\,\%3\,c^3\,t1^2\,t2}{\%1}$$

$$-\frac{4\,\%3\,c^3\,t1\,t2^2}{\%1} + \frac{16\,\%3\,t1\,c^3\,t3^2}{\%1} - \frac{13\,\%3\,c^3\,t1^2\,t3}{\%1} - 62\,c^4\,t1^3\,t3 + \frac{\%3\,c\,t1}{\%1}$$

$$+\frac{2\,\%3\,c^3\,t1^3}{\%1} - \frac{5\,\%3\,c^3\,t3^3}{\%1} - 26\,c^4\,t1^3\,t2 + \frac{4\,\%3\,c\,t2}{\%1} - \frac{5\,\%3\,c\,t3}{\%1}$$

$$+10\,t3^2\,c^4\,t2^2 - 30\,t1\,c^4\,t3^3 + 70\,c^4\,t1^2\,t3^2 - 12\,c^4\,t2^3\,t1 + 16\,c^4\,t1^2\,t2^2 + 5\,c^4\,t3^4$$

$$-18\,c^2\,t2^2 + 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2$$

$$+22\,c^4\,t1^4 + 5\,c^4\,t2^4)/(4\,\%1), -(50 + 26\,c^2\,t1\,t2 - 8\,t3\,c^4\,t2^3 - 30\,t1\,c^2\,t3$$

$$+10\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2 + \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1} - \frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1}$$

$$-\frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1} - \frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1}$$

$$-62\,c^4\,t1^3\,t3 - 26\,c^4\,t1^3\,t2 + 10\,t3^2\,c^4\,t2^2 - \frac{5\,\%4\,c^3\,t3^3}{\%1} + \frac{4\,\%4\,c\,t2}{\%1} - \frac{5\,\%4\,c\,t3}{\%1}$$

$$+\frac{\%4\,c\,t1}{\%1} + \frac{2\,\%4\,c^3\,t1^3}{\%1} - 30\,t1\,c^4\,t3^3 + 70\,c^4\,t1^2\,t3^2 - 12\,c^4\,t2^3\,t1 + 16\,c^4\,t1^2\,t2^2$$

$$+5\,c^4\,t3^4 - 18\,c^2\,t2^2 + 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2$$

$$+10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/(4\,\%1)) + c^2\,t1^2 - 2\,c^2\,t1\,t2 + c^2\,t2^2)^{(1/2)}$$

$$\%1 := 2\,c^2\,t1^2 + 5\,c^2\,t3^2 + 5\,c^2\,t2^2 - 9 - 8\,t3\,c^2\,t2 - 2\,t1\,c^2\,t3 - 2\,c^2\,t1\,t2$$

$$\%2 := 50 + 20\,c^2\,t1\,t2 - 10\,t3\,c^4\,t2^3 - 20\,t1\,c^2\,t3 + 50\,t3\,c^2\,t2 + 10\,t3^3\,c^4\,t2 - 56\,c^4\,t1^3\,t3$$

$$-16\,c^4\,t1^3\,t2 + 8\,t3^2\,c^4\,t2^2 - 30\,t1\,c^4\,t3^3 + 68\,c^4\,t1^2\,t3^2 - 10\,c^4\,t2^3\,t1 + 8\,c^4\,t1^2\,t2^2$$

$$+5\,c^4\,t3^4 - 35\,c^2\,t2^2 + 32\,c^4\,t1^2\,t3\,t2 + 14\,c^4\,t1\,t2^2\,t3 - 46\,c^4\,t1\,t2\,t3^2 - 15\,c^2\,t3^2$$

$$+18\,c^4\,t1^4 + 5\,c^4\,t2^4 - 4\,c^6\,t1^6 - 34\,c^6\,t1^4\,t3\,t2 + 34\,c^6\,t1^3\,t3\,t2^2 - 14\,c^6\,t1^2\,t3^3\,t2$$

$$-30\,c^6\,t1^2\,t3^2\,t2^2 + 34\,c^6\,t1^3\,t3^2\,t2 - 14\,c^6\,t1^2\,t2^3\,t3 + 6\,c^6\,t1^3\,t3^3 + 12\,c^6\,t1^5\,t3$$

$$-c^6\,t1^2\,t3^4 - 13\,c^6\,t1^4\,t2^2 + 6\,c^6\,t1^3\,t2^3 + 12\,c^6\,t1^5\,t2 - c^6\,t1^2\,t2^4 - t3^4\,c^6\,t2^2$$

$$-2\,t3^3\,c^6\,t2^3 - t3^2\,c^6\,t2^4 - 13\,c^6\,t1^4\,t3^2 + 10\,c^6\,t1\,t2^3\,t3^2 + 2\,c^6\,t1\,t2^4\,t3$$

$$+10\,c^6\,t1\,t2^2\,t3^3 + 2\,t1\,c^6\,t3^4\,t2$$

$$\%3 := 26\,c^3\,t1^2\,t3 - 14\,c^3\,t1^2\,t2 + 8\,c^3\,t1\,t2^2 - 32\,t1\,c^3\,t3^2 + 2\,t3^2\,c^3\,t2 - 8\,t3\,c^3\,t2^2$$

$$-4\,c^3\,t1^3 + 10\,c^3\,t3^3 + 10\,c\,t3 - 8\,c\,t2 - 2\,c\,t1 + 12\,c^3\,t1\,t2\,t3 + 6\,\sqrt{\%2}$$

$$\%4 := 26\,c^3\,t1^2\,t3 - 14\,c^3\,t1^2\,t2 + 8\,c^3\,t1\,t2^2 - 32\,t1\,c^3\,t3^2 + 2\,t3^2\,c^3\,t2 - 8\,t3\,c^3\,t2^2$$

$$- 4\,c^3\,t1^3 + 10\,c^3\,t3^3 + 10\,c\,t3 - 8\,c\,t2 - 2\,c\,t1 + 12\,c^3\,t1\,t2\,t3 - 6\sqrt{\%2}$$

## 27.   Assembly Calculation Sequence For A Target Position Report

The order of arithmetic steps to calculate a Target Position Report, given arrival times $t_1, t_2, t_3$, does not depend on the coefficients found in a particular MAPLE calculation. In fact, a generic calculation sequence is available regardless of the values assigned to the $(x_i, y_i)$. For the purposes of this section only, the coefficient and $p_i$ values for the $X$-value of Target Position Report Example #1 shall be used as a demonstration of the calculation sequence that may be utilized to assemble a Target Position Report. This sequence is oriented to the anticipated MMIX assembly language implementation that forms the PQIC OSMMTS instantiation.

$$\%1 := 5\,c^2\,t2^2 + 2\,c^2\,t1^2 + 5\,c^2\,t3^2 - 9 - 2\,t1\,c^2\,t3 - 2\,c^2\,t1\,t2 - 8\,t3\,c^2\,t2$$

$$
\begin{aligned}
\%2 := \; & 50 - c^6\,t1^2\,t2^4 + 12\,c^6\,t1^5\,t2 + 6\,c^6\,t1^3\,t2^3 - 13\,c^6\,t1^4\,t2^2 + 12\,c^6\,t1^5\,t3 - c^6\,t1^2\,t3^4 \\
& + 6\,c^6\,t1^3\,t3^3 - 13\,c^6\,t1^4\,t3^2 - 16\,c^4\,t1^3\,t2 - 14\,c^6\,t1^2\,t3^3\,t2 + 34\,c^6\,t1^3\,t3\,t2^2 \\
& - 34\,c^6\,t1^4\,t3\,t2 - t3^2\,c^6\,t2^4 - 2\,t3^3\,c^6\,t2^3 - t3^4\,c^6\,t2^2 + 34\,c^6\,t1^3\,t3^2\,t2 \\
& - 30\,t1\,c^4\,t3^3 + 8\,c^4\,t1^2\,t2^2 - 30\,c^6\,t1^2\,t3^2\,t2^2 - 20\,t1\,c^2\,t3 - 10\,c^4\,t2^3\,t1 \\
& - 14\,c^6\,t1^2\,t2^3\,t3 + 20\,c^2\,t1\,t2 + 10\,c^6\,t1\,t2^2\,t3^3 + 2\,c^6\,t1\,t2^4\,t3 + 10\,c^6\,t1\,t2^3\,t3^2 \\
& + 68\,c^4\,t1^2\,t3^2 + 2\,t1\,c^6\,t3^4\,t2 + 8\,t3^2\,c^4\,t2^2 + 50\,t3\,c^2\,t2 - 10\,t3\,c^4\,t2^3 - 4\,c^6\,t1^6 \\
& - 56\,c^4\,t1^3\,t3 + 10\,t3^3\,c^4\,t2 + 5\,c^4\,t3^4 - 35\,c^2\,t2^2 + 32\,c^4\,t1^2\,t3\,t2 + 14\,c^4\,t1\,t2^2\,t3 \\
& - 46\,c^4\,t1\,t2\,t3^2 - 15\,c^2\,t3^2 + 18\,c^4\,t1^4 + 5\,c^4\,t2^4
\end{aligned}
$$

$$
\begin{aligned}
\%3 := \; & 26\,c^3\,t1^2\,t3 - 14\,c^3\,t1^2\,t2 + 8\,c^3\,t1\,t2^2 - 32\,t1\,c^3\,t3^2 + 2\,t3^2\,c^3\,t2 - 8\,t3\,c^3\,t2^2 \\
& - 8\,c\,t2 + 10\,c^3\,t3^3 - 2\,c\,t1 + 10\,c\,t3 + 12\,c^3\,t1\,t2\,t3 - 4\,c^3\,t1^3 - 6\sqrt{\%2}
\end{aligned}
$$

$$\%4 := \%3 + 12\sqrt{\%2}$$

$$
\begin{aligned}
\#1 := \; & -\Big(50 - \frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - 26\,c^4\,t1^3\,t2 - 30\,t1\,c^4\,t3^3 \\
& + 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3 - 12\,c^4\,t2^3\,t1 + 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2 \\
& + 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3 - \frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1} \\
& - \frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1} - 62\,c^4\,t1^3\,t3 - \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1} + 10\,t3^3\,c^4\,t2
\end{aligned}
$$

$$+ \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1} + 5\,c^4\,t3^4 - \frac{5\,\%4\,c\,t3}{\%1} + \frac{\%4\,c\,t1}{\%1} - \frac{5\,\%4\,c^3\,t3^3}{\%1} + \frac{4\,\%4\,c\,t2}{\%1}$$

$$+ \frac{2\,\%4\,c^3\,t1^3}{\%1} - 18\,c^2\,t2^2 + 2\,c^2\,t1^2 + 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3$$

$$- 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/(\%1)$$

$$\#2 := -(50 - 26\,c^4\,t1^3\,t2$$

$$- 30\,t1\,c^4\,t3^3 + 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3 - 12\,c^4\,t2^3\,t1 + 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2$$

$$+ 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3 - 62\,c^4\,t1^3\,t3 + 10\,t3^3\,c^4\,t2$$

$$- \frac{4\,\%3\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%3\,c^3\,t1^2\,t2}{\%1} - \frac{13\,\%3\,c^3\,t1^2\,t3}{\%1} - \frac{6\,\%3\,c^3\,t1\,t2\,t3}{\%1}$$

$$+ \frac{16\,\%3\,t1\,c^3\,t3^2}{\%1} + \frac{4\,\%3\,t3\,c^3\,t2^2}{\%1} - \frac{\%3\,t3^2\,c^3\,t2}{\%1} + 5\,c^4\,t3^4 + \frac{\%3\,c\,t1}{\%1}$$

$$- \frac{5\,\%3\,c^3\,t3^3}{\%1} + \frac{4\,\%3\,c\,t2}{\%1} + \frac{2\,\%3\,c^3\,t1^3}{\%1} - \frac{5\,\%3\,c\,t3}{\%1} - 18\,c^2\,t2^2 + 2\,c^2\,t1^2$$

$$+ 46\,c^4\,t1^2\,t3\,t2 + 4\,c^4\,t1\,t2^2\,t3 - 50\,c^4\,t1\,t2\,t3^2 + 10\,c^2\,t3^2 + 22\,c^4\,t1^4 + 5\,c^4\,t2^4)/(\\ \%1)$$

$$\#3 := c^2\,t1^2 - 2\,c^2\,t1\,t2 + c^2\,t2^2$$

$$\#0 := (2\,t1 - t2 - t3)\,c$$

$$\#00 := -\frac{1}{6}(t1 - t3)\,c$$

$$A := \left(\#0 + \sqrt{\#1 + \#2 + \#3}\right) \times \#00$$

$$\#4 := \frac{17}{6}$$

$$\#5 := \frac{1}{3}(t1 - t2)\,c$$

$$\#6 := -(50 - \frac{13\,\%4\,c^3\,t1^2\,t3}{\%1} - 26\,c^4\,t1^3\,t2 - 30\,t1\,c^4\,t3^3 + 16\,c^4\,t1^2\,t2^2 - 30\,t1\,c^2\,t3$$

$$- 12\,c^4\,t2^3\,t1 + 26\,c^2\,t1\,t2 + 70\,c^4\,t1^2\,t3^2 + 10\,t3^2\,c^4\,t2^2 + 10\,t3\,c^2\,t2 - 8\,t3\,c^4\,t2^3$$

$$- \frac{\%4\,t3^2\,c^3\,t2}{\%1} + \frac{16\,\%4\,t1\,c^3\,t3^2}{\%1} - \frac{4\,\%4\,c^3\,t1\,t2^2}{\%1} + \frac{7\,\%4\,c^3\,t1^2\,t2}{\%1} - 62\,c^4\,t1^3\,t3$$

$$- \frac{6\,\%4\,c^3\,t1\,t2\,t3}{\%1} + 10\,t3^3\,c^4\,t2 + \frac{4\,\%4\,t3\,c^3\,t2^2}{\%1} + 5\,c^4\,t3^4 - \frac{5\,\%4\,c\,t3}{\%1}$$

$$+ \frac{\%4\, c\, t1}{\%1} - \frac{5\,\%4\, c^3\, t3^3}{\%1} + \frac{4\,\%4\, c\, t2}{\%1} + \frac{2\,\%4\, c^3\, t1^3}{\%1} - 18\, c^2\, t2^2 + 2\, c^2\, t1^2$$

$$+ 46\, c^4\, t1^2\, t3\, t2 + 4\, c^4\, t1\, t2^2\, t3 - 50\, c^4\, t1\, t2\, t3^2 + 10\, c^2\, t3^2 + 22\, c^4\, t1^4 + 5\, c^4\, t2^4)/(\%1)$$

$$\#7 := -(50 - 26\, c^4\, t1^3\, t2 - 30\, t1\, c^4\, t3^3 + 16\, c^4\, t1^2\, t2^2 - 30\, t1\, c^2\, t3 - 12\, c^4\, t2^3\, t1$$

$$+ 26\, c^2\, t1\, t2 + 70\, c^4\, t1^2\, t3^2 + 10\, t3^2\, c^4\, t2^2 + 10\, t3\, c^2\, t2 - 8\, t3\, c^4\, t2^3 - 62\, c^4\, t1^3\, t3$$

$$+ 10\, t3^3\, c^4\, t2 - \frac{4\,\%3\, c^3\, t1\, t2^2}{\%1} + \frac{7\,\%3\, c^3\, t1^2\, t2}{\%1} - \frac{13\,\%3\, c^3\, t1^2\, t3}{\%1}$$

$$- \frac{6\,\%3\, c^3\, t1\, t2\, t3}{\%1} + \frac{16\,\%3\, t1\, c^3\, t3^2}{\%1} + \frac{4\,\%3\, t3\, c^3\, t2^2}{\%1} - \frac{\%3\, t3^2\, c^3\, t2}{\%1} + 5\, c^4\, t3^4$$

$$+ \frac{\%3\, c\, t1}{\%1} - \frac{5\,\%3\, c^3\, t3^3}{\%1} + \frac{4\,\%3\, c\, t2}{\%1} + \frac{2\,\%3\, c^3\, t1^3}{\%1} - \frac{5\,\%3\, c\, t3}{\%1} - 18\, c^2\, t2^2$$

$$+ 2\, c^2\, t1^2 + 46\, c^4\, t1^2\, t3\, t2 + 4\, c^4\, t1\, t2^2\, t3 - 50\, c^4\, t1\, t2\, t3^2 + 10\, c^2\, t3^2 + 22\, c^4\, t1^4$$

$$+ 5\, c^4\, t2^4)/(\%1)$$

$$\#8 := c^2\, t1^2 - 2\, c^2\, t1\, t2 + c^2\, t2^2$$

$$B := \left(\#5 \times \sqrt{\#6 + \#7 + \#8}\right) + \#4$$

$$x_0 := A + B$$

## 28. Error Distribution

All data collected during process actions are subject to error. The source or cause of the error varies by implementation, however, the OSMMTS, like all such processes, has several dominant sources of data error. There is *mechanical error* introduced by the physical limitations of the mechanical devices to precisely calculate the exact time of signal detection. There is *propagation error* from the natural corruption of the signal before detection. There is also *rounding error* due to the finite precision calculations utilized in the implementation, and *algorithm error* introduced by inefficient numerical methods.

The PQIC OSMMTS may be used to specifically addresses rounding and algorithm errors by framing the implementation of all algorithms in assembly code, e.g., within a system that uses the MMIX[11] instruction set and memory model, even though this approach is not required in any particular OSMMTS implementation[12]. However, utilizing MMIX-based assembly code may be considered the OSMMTS "preferred embodiment," with regard to algorithm implementation, since all non-mechanical and non-propagation error issues are addressed by this approach.

Finally, propagation error may be addressed, if not eliminated, by the optimal placement and mitigations for reflections methods documented in this memorandum. Mechanical errors may be addressed by the merchantability of the PCPU.

The primary focus of the error distribution calculations is to account for the remaining effect of error in the arrival time data, regardless of source, after all precautions, optimizations, and mitigations have been used to minimize said error.

**Definition 8** *The* MLTarget Position *is the calculated target position most likely to be correct given the arrival time data and the variability of the arrival times. It is found at the same time the* $M_0$ *distribution is calculated (see 9).*

## 29. Derivation

Suppose $t_1$ and $t_2$ are not precisely measured, in the sense that

$$t_1^* = t_1 + \varepsilon_1 \text{ and } t_2^* = t_2 + \varepsilon_2$$

where the $t_i^*$ are the observed values, and the $t_i$ are the actual "true" times. Here, each $\varepsilon_i$ is an error term. Suppose each $\varepsilon_i \sim N\left(0, \sigma^2\right)$, for a common $\sigma > 0$, and all such $\varepsilon_i$ are IID. Then

$$
\begin{aligned}
f^* &= t_1^* - t_2^* \\
&= (t_1 + \varepsilon_1) - (t_2 + \varepsilon_2) \\
&= (t_1 - t_2) + (\varepsilon_1 - \varepsilon_2) \\
&= f + (\varepsilon_1 - \varepsilon_2)
\end{aligned}
$$

which means

$$f^* - f = \varepsilon_1 - \varepsilon_2 \sim N\left(0, 2\sigma^2\right)$$

Then

$$s_1^* - s_2^* = f^*c \text{ and } s_1 - s_2 = fc$$

mean

$$(s_1^* - s_2^*) - (s_1 - s_2) = (f^* - f)c \sim N\left(0, 2c^2\sigma^2\right)$$

or

$$(s_{21}^* - s_{11}^*) - (s_{21} - s_{11}) = -(f_1^* - f_1)c \sim N\left(0, 2c^2\sigma^2\right)$$

$$(s_{22}^* - s_{12}^*) - (s_{22} - s_{21}) = -(f_2^* - f_2)c \sim N\left(0, 2c^2\sigma^2\right)$$

Now from (1), we have

$$d_1^2 = s_{11}^2 + s_{21}^2 - 2s_{11}s_{21}\cos\theta_0$$

and

$$d_1^2 = s_{11}^{*2} + s_{21}^{*2} - 2s_{11}^* s_{21}^* \cos\theta_0^*$$

---

[11] MMIX (2009): A RISC Computer For The Third Millennium, as developed by Dr. Donald E. Knuth of Stanford University, is documented at http://www-cs-faculty.stanford.edu/~knuth/mmix.html.

[12] An OSMMTS implementation will work as well without MMIX as any other implemented OSMMTS as long as the error issues identified herein are addressed.

so

$$s_{11}^2 + s_{21}^2 - 2\kappa_0 \cos\theta_0 = s_{11}^{*2} + s_{21}^{*2} - 2\kappa_0^* \cos\theta_0^*$$

However,

$$
\begin{aligned}
s_{11}^2 + s_{21}^2 &= \left(\tfrac{1}{2}\left(f_1 c \pm \sqrt{4\kappa_0 + (f_1 c)^2}\right)\right)^2 + \left(\tfrac{1}{2}\left(-f_1 c \pm \sqrt{4\kappa_0 + (f_1 c)^2}\right)\right)^2 \\
&= \tfrac{1}{4}\left(4(f_1 c)^2 + 8\kappa_0\right) \\
&= (f_1 c)^2 + 2\kappa_0
\end{aligned}
$$

and

$$s_{11}^{*2} + s_{21}^{*2} = (f_1^* c)^2 + 2\kappa_0^*$$

Therefore,

$$(f_1 c)^2 + 2\kappa_0 (1 - \cos\theta_0) = (f_1^* c)^2 + 2\kappa_0^* (1 - \cos\theta_0^*)$$

so

$$\kappa_0^* (1 - \cos\theta_0^*) - \kappa_0 (1 - \cos\theta_0) = \frac{1}{2}\left(((f_1^* - (\varepsilon_1 - \varepsilon_2))c)^2 - (f_1^* c)^2\right) \tag{12}$$

These calculations lead to the first important subsidiary distribution for quantifying the OSMMTS error.

**Definition 9** Let $\eta = \eta(t_1^*, t_2^*, t_3^*, \varepsilon_1, \varepsilon_2, \varepsilon_3) = \cos\theta_0^* - \cos\theta_0$. Then $\eta$ is said to have an $M_0(t_1^*, t_2^*, t_3^*)$ distribution.

Hence,

$$1 - \cos\theta_0 = 1 - \cos\theta_0^* + \eta$$

and

$$
\begin{aligned}
\left(s_{21}^{*2} - s_{11}^{*2}\right) - \left(s_{21}^2 - s_{11}^2\right) &= \mp f_1^* c\sqrt{4\kappa_0^* + (f_1^* c)^2} \pm f_1 c\sqrt{4\kappa_0 + (f_1 c)^2} \\
&= \mp f_1^* c\sqrt{4\kappa_0^* + (f_1^* c)^2} \pm f_1 c\sqrt{2\left(\frac{d_1^2 - (f_1 c)^2}{(1 - \cos\theta_0)}\right) + (f_1 c)^2} \\
&= \begin{cases} \mp f_1^* c\sqrt{4\kappa_0^* + (f_1^* c)^2} \\ \pm (f_1^* - (\varepsilon_1 - \varepsilon_2))c\sqrt{2\left(\frac{d_1^2 - ((f_1^* - (\varepsilon_1 - \varepsilon_2))c)^2}{(1 - \cos\theta_0^* + \eta)}\right) + ((f_1^* - (\varepsilon_1 - \varepsilon_2))c)^2} \end{cases}
\end{aligned}
$$

If $\xi_1 = (f_1^* - (\varepsilon_1 - \varepsilon_2))c \sim N\left(f_1^* c, 2c^2\sigma^2\right)$, then we have

$$\left(s_{21}^{*2} - s_{11}^{*2}\right) - \left(s_{21}^2 - s_{11}^2\right) = \mp f_1^* c\sqrt{2\left(\frac{d_1^2 - (f_1^* c)^2}{(1 - \cos\theta_0^*)}\right) + (f_1^* c)^2} \pm \xi_1\sqrt{2\left(\frac{d_1^2 - \xi_1^2}{(1 - \cos\theta_0^* + \eta)}\right) + \xi_1^2} \tag{13}$$

**Definition 10** When $\xi \sim N\left(f_1^* c, 2c^2\sigma^2\right)$ and $\eta \sim M_0(t_1^*, t_2^*, t_3^*)$, then

$$\frac{1}{2(y_2 - y_1)}\left(\mp f_1^* c\sqrt{2\left(\frac{d_1^2 - (f_1^* c)^2}{(1 - \cos\theta_0^*)}\right) + (f_1^* c)^2} \pm \xi_1\sqrt{2\left(\frac{d_1^2 - \xi_1^2}{(1 - \cos\theta_0^* + \eta)}\right) + \xi_1^2}\right)$$

is distributed according to the $M_1(t_1^*, t_2^*, t_3^*)$ probability distribution function.

We also have that when $\xi_2 = (f_2^* - (\varepsilon_1 - \varepsilon_3))c \sim N\left(f_2^* c, 2c^2\sigma^2\right)$ and $\eta \sim M_0(t_1^*, t_2^*, t_3^*)$, then

$$\frac{1}{2(y_3 - y_1)}\left(\mp f_2^* c\sqrt{2\left(\frac{d_2^2 - (f_2^* c)^2}{(1 - \cos\theta_0^*)}\right) + (f_2^* c)^2} \pm \xi_2\sqrt{2\left(\frac{d_2^2 - \xi_2^2}{(1 - \cos\theta_0^* + \eta)}\right) + \xi_2^2}\right) \sim M_1(t_1, t_2, t_3)$$

**Definition 11** If $X \sim M_1(t_1^*, t_2^*, t_3^*)$ and $Y \sim M_1(t_1^*, t_2^*, t_3^*)$, and $X$ and $Y$ are independent, then

$$X - Y \sim M_2(t_1^*, t_2^*, t_3^*)$$

**Definition 12** If $J \sim M_2(t_1^*, t_2^*, t_3^*)$ and $K \sim M_2(t_1^*, t_2^*, t_3^*)$, and $J$ and $K$ are independent, then

$$(m_1 - m_2) J^2 + \left(1 - \frac{m_2}{m_1}\right) K^2 \sim M_3(t_1^*, t_2^*, t_3^*)$$

By (13), we have

$$\frac{\left(s_{21}^{*2} - s_{11}^{*2}\right) - \left(s_{21}^2 - s_{11}^2\right)}{2\left(y_2 - y_1\right)} \sim M_1\left(t_1^*, t_2^*, t_3^*\right)$$

with a similar statement for $\frac{\left(s_{21}^{*2} - s_{11}^{*2}\right) - \left(s_{21}^2 - s_{11}^2\right)}{2\left(y_3 - y_1\right)} \sim M_1\left(t_1^*, t_2^*, t_3^*\right)$. Let

$$\left(x_0^*, y_0^*\right) = \left(\frac{b_2^* - b_1^*}{m_1 - m_2}, m_1\left(\frac{b_2^* - b_1^*}{m_1 - m_2}\right) + b_1^*\right)$$

with similar results for other forms of $\left(x_0^*, y_0^*\right)$ corresponding to $\left(x_0, y_0\right)$, where

$$b_1^* = \frac{s_{21}^{*2} - s_{11}^{*2} + \left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right)}{2\left(y_2 - y_1\right)}$$

$$b_2^* = \frac{s_{22}^{*2} - s_{12}^{*2} + \left(x_3^2 - x_1^2\right) + \left(y_3^2 - y_1^2\right)}{2\left(y_3 - y_1\right)}$$

Note how the value of $m_1$ and $m_2$ are unaffected by consideration of the $t_i^*$.

Then

$$b_1^* = \frac{s_{21}^2 - s_{11}^2 + \left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right)}{2\left(y_2 - y_1\right)} + \frac{\left(s_{21}^{*2} - s_{11}^{*2}\right) - \left(s_{21}^2 - s_{11}^2\right)}{2\left(y_2 - y_1\right)}$$

$$b_2^* = \frac{s_{22}^2 - s_{12}^2 + \left(x_3^2 - x_1^2\right) + \left(y_3^2 - y_1^2\right)}{2\left(y_3 - y_1\right)} + \frac{\left(s_{22}^{*2} - s_{12}^{*2}\right) - \left(s_{22}^2 - s_{12}^2\right)}{2\left(y_3 - y_1\right)}$$

or

$$b_1^* = b_1 + \frac{\left(s_{21}^{*2} - s_{11}^{*2}\right) - \left(s_{21}^2 - s_{11}^2\right)}{2\left(y_2 - y_1\right)}$$

$$b_2^* = b_2 + \frac{\left(s_{22}^{*2} - s_{12}^{*2}\right) - \left(s_{22}^2 - s_{12}^2\right)}{2\left(y_3 - y_1\right)}$$

By Definition 10, this means

$$b_1^* - b_1 \sim M_1\left(t_1^*, t_2^*, t_3^*\right)$$

$$b_2^* - b_2 \sim M_1\left(t_1^*, t_2^*, t_3^*\right)$$

Therefore,

$$\begin{aligned}
x_0^* - x_0 &= \frac{b_2^* - b_1^*}{m_1 - m_2} - \frac{b_2 - b_1}{m_1 - m_2} \\
&= \frac{\left(b_2^* - b_1^*\right) - \left(b_2 - b_1\right)}{m_1 - m_2} \\
&= \frac{\left(b_2^* - b_2\right) - \left(b_1^* - b_1\right)}{m_1 - m_2}
\end{aligned}$$

and by Definition 11, we have

$$\left(m_1 - m_2\right)\left(x_0^* - x_0\right) \sim M_2\left(t_1^*, t_2^*, t_3^*\right)$$

Likewise,

$$\begin{aligned}
y_0^* - y_0 &= m_1\left(\frac{b_2^* - b_1^*}{m_1 - m_2}\right) + b_1 - \left(m_1\left(\frac{b_2 - b_1}{m_1 - m_2}\right) + b_1\right) \\
&= m_1\left(\frac{b_2^* - b_1^*}{m_1 - m_2} - \frac{b_2 - b_1}{m_1 - m_2}\right) \\
&= \frac{m_1}{m_1 - m_2}\left(\left(b_2^* - b_2\right) - \left(b_1^* - b_1\right)\right)
\end{aligned}$$

and by Definition 11, we have

$$\left(1 - \frac{m_2}{m_1}\right)\left(y_0^* - y_0\right) \sim M_2\left(t_1^*, t_2^*, t_3^*\right)$$

## 30. The Error Likelihood Ellipse

**Definition 13** *The* Error Likelihood Ellipse *(ELE) for the OSMMTS position* $\left(x_0, y_0\right)$ *is*

$$\left(\frac{x_0^* - x_0}{1 - \frac{m_2}{m_1}}\right)^2 + \left(\frac{y_0^* - y_0}{m_1 - m_2}\right)^2 = R$$

*with* $\left(m_1 - m_2\right)\left(x_0^* - x_0\right) \sim M_2\left(t_1^*, t_2^*, t_3^*\right)$ *and* $\left(1 - \frac{m_2}{m_1}\right)\left(y_0^* - y_0\right) \sim M_2\left(t_1^*, t_2^*, t_3^*\right)$, *and where the constant* $R$ *depends on the two* $M_2\left(t_1^*, t_2^*, t_3^*\right)$ *distributions.*

However, we have

$$
\begin{aligned}
R &= \left(\frac{x_0^* - x_0}{\sqrt{1 - \frac{m_2}{m_1}}}\right)^2 + \left(\frac{y_0^* - y_0}{\sqrt{m_1 - m_2}}\right)^2 \\
&= \frac{1}{1 - \frac{m_2}{m_1}}\left(x_0^* - x_0\right)^2 + \frac{1}{m_1 - m_2}\left(y_0^* - y_0\right)^2 \\
&= \frac{(m_1 - m_2)(x_0^* - x_0)^2 + \left(1 - \frac{m_2}{m_1}\right)(y_0^* - y_0)^2}{(m_1 - m_2)\left(1 - \frac{m_2}{m_1}\right)} \\
&= \frac{(m_1 - m_2)(x_0^* - x_0)^2 + \left(1 - \frac{m_2}{m_1}\right)(y_0^* - y_0)^2}{\left(\frac{(m_1 - m_2)^2}{m_1}\right)}
\end{aligned}
$$

or

$$
(m_1 - m_2)\left(x_0^* - x_0\right)^2 + \left(1 - \frac{m_2}{m_1}\right)\left(y_0^* - y_0\right)^2 = \frac{(m_1 - m_2)^2}{m_1} R
$$

By Definitions 11 and 12, we have

$$
\frac{(m_1 - m_2)^2}{m_1} R \sim M_3\left(t_1^*, t_2^*, t_3^*\right)
$$

Hence, the likelihood of the Target $\mathbf{T}$ being within the Error Likelihood Ellipse with constant $\frac{(m_1 - m_2)^2}{m_1} R$ is calculated by the probabilities of $M_3\left(t_1^*, t_2^*, t_3^*\right)$.

## 31.  Error Likelihood Ellipse Calculation

The Error Likelihood Ellipse is the totality of Target Positions Reports for a particular set of three SDU's produced by varying the observed arrival time data according to an assigned error distribution. The resulting $(x, y)$ position that has maximum likelihood within this ellipse is judged to be the most likely "real" position of the target. The same algorithms used to calculate an individual Target Position Report shall be used to calculate the Error Likelihood Ellipse.

## 32.  Error Likelihood Ellipse Standard Methodology

The PQIC OSMMTS Error Likelihood Ellipse Standard Methodology is to assume the arrival time data varies according to a normal distribution of mean zero and standard deviation $\sigma$, where $\sigma$ is either estimated from calibration data or the result of a Bayesian estimation process. This means the observed arrival time data is always accepted as the center of the error distribution.

The algorithms found below implement such an error distribution. However, such algorithms may be amended at any time to more accurately reflect the true error distribution in any particular PQIC OSMMTS System implementation.

## 33.  Error Likelihood Ellipse MAPLE Algorithm

Given the position information of the SDU's that receive the arrival time data, the following MAPLE algorithm calculates the Error Likelihood Ellipse based on the normal error distribution affected arrival times. These algorithms call the Target Position Report algorithms by reference, so that the Error Likelihood Ellipse algorithms are independent of the particular methods found in the Target Position Report code.

The $M_0$ distribution and the Target Position Report algorithms are needed to calculate the Error Likelihood Ellipse. All other $M$ distributions are only needed when quantifying the bounds of the error ellipse, or for making multivariate calculations on multiple Target Position Reports and Error Likelihood Ellipses.

**34.**  $\langle$ ELE Calculation 34 $\rangle \equiv$
  MOGEN: $= proc(t1, t2, t3, a, b, r)$
  *local ans*, $e1$, $e2$, $e3$, $j1$, $j2$, $j3$, $idx$, $tx$;
  *options* '*Copyright* 2003 PQI *Consulting All Rights Reserved*';
  *description* "OSMMTS␣M0␣Generation␣Calculation";
  $idx$: $= 0$;
  *with*(*stats*);
  **for** $e1$ *from* $a$ *to* $b$ **do** $j1$: $= statevalf[cdf, normald[0, 0.1]](e1/r) - statevalf[cdf, normald[0, 0.1]]((e1-1)/r)$;
  **for** $e2$ *from* $a$ *to* $b$ **do** $j2$: $= statevalf[cdf, normald[0, 0.1]](e2/r) - statevalf[cdf, normald[0, 0.1]]((e2-1)/r)$;
  **for** $e3$ *from* $a$ *to* $b$ **do** $j3$: $= statevalf[cdf, normald[0, 0.1]](e3/r) - statevalf[cdf, normald[0, 0.1]]((e3-1)/r)$;
  $tx$: $= evalf(osmmts(t1 + e1/r, t2 + e2/r, t3 + e3/r)[], 5)$;
  **if** $(type(tx[3], undefined))$ **then**
  **else**
  **if** $(abs(tx[4]) > 1)$ **then**
  **else**
  $idx$: $= idx + 1$;
  $ans[idx]$: $= (tx[1], tx[2], j1 * j2 * j3)$;
  **end if** ;
  **end if** ;
  **end do** ;
  **end do** ;
  **end do** ;
  **return** $ans$, $idx$;
  *end proc*;

$J:\ =\ proc(dt, cnt, msh)$

$local\ idx, inx, iny, \mathbf{new}, smm, minx, maxx, miny, maxy, g, gx, gy, M;$

$options\ `Copyright\ 2003\ PQI\ Consulting\ All\ Rights\ Reserved`;$

$description\ "\texttt{OSMMTS\_MO\_Processing}";$

$minx:\ =\ 0;\ maxx:\ =\ 0;$

$miny:\ =\ 0;\ maxy:\ =\ 0;$

$g:\ =\ 0;$

$\mathbf{for}\ idx\ \mathbf{from}\ 1\ \mathbf{to}\ cnt\ \mathbf{do}\ \mathbf{new}:\ =\ dt[idx];$

$inx:\ =\ floor(msh * \mathbf{new}[1]);$

$iny:\ =\ floor(msh * \mathbf{new}[2]);$

$\mathbf{if}\ (assigned(M[inx, iny]))\ \mathbf{then}$

$M[inx, iny]:\ =\ M[inx, iny] + \mathbf{new}[3];$

$\mathbf{else}\ M[inx, iny]:\ =\ \mathbf{new}[3];$

$\mathbf{end\ if}\ ;$

$\mathbf{if}\ (inx > maxx)\ \mathbf{then}$

$maxx:\ =\ inx;$

$\mathbf{end\ if}\ ;$

$\mathbf{if}\ (inx < minx)\ \mathbf{then}$

$minx:\ =\ inx;$

$\mathbf{end\ if}\ ;$

$\mathbf{if}\ (iny > maxy)\ \mathbf{then}$

$maxy:\ =\ inx;$

$\mathbf{end\ if}\ ;$

$\mathbf{if}\ (iny < miny)\ \mathbf{then}$

$miny:\ =\ inx;$

$\mathbf{end\ if}\ ;$

$\mathbf{end\ do}\ ;$

$\mathbf{for}\ idx\ \mathbf{from}\ minx\ \mathbf{to}\ maxx\ \mathbf{do}\ \mathbf{for}\ iny\ \mathbf{from}\ miny\ \mathbf{to}\ maxy\ \mathbf{do}$

$\mathbf{if}\ (assigned(M[idx, iny]))\ \mathbf{then}$

$\mathbf{if}\ (M[idx, iny] > g)\ \mathbf{then}$

$g:\ =\ M[idx, iny];$

$gx:\ =\ idx;$

$gy:\ =\ iny;$

$\mathbf{end\ if}\ ;$

$\mathbf{else}\ M[idx, iny]:\ =\ 0;$

$\mathbf{end\ if}\ ;$

$\mathbf{end\ do}\ ;$

$\mathbf{end\ do}\ ;$

$\mathbf{return}\ eval(M), minx, maxx, miny, maxy, gx, gy, g;$

$\mathbf{end\ proc};$

This code is used in chunk 12.

## 35. Error Likelihood Ellipse Example #1

**36.** ⟨ ELE Example #1 SDU Coordinates 36 ⟩ ≡

$(x1, y1):\ =\ (2, 3);\ (x2, y2):\ =\ (4, 2);\ (x3, y3):\ =\ (3, 1);$

This code is used in chunk 12.

## 37. Error Likelihood Ellipse MAPLE Calculation For Example #1

```
>   with(stats):
>   mesh:=35;
>   TPR:=(evalf(osmmts(16/10,2,175/100)[1..2],10));
```

$TPR := 2.547503694, 2.097308261$

```
> RES:=MOGEN(16/10,2,175/100,-12,12,mesh):
```

```
Warning, these names have been redefined: anova, describe, fit,
importdata, random, statevalf, statplots, transform
```

```
> MZ:=J(RES[1],RES[2],mesh):
```

## 38. Error Likelihood Ellipse Example #2

## 39. ⟨ ELE Example #2 SDU Coordinates 39 ⟩ ≡
$(x1, y1)$: $= (3, 2)$; $(x2, y2)$: $= (2, 4)$; $(x3, y3)$: $= (1, 3)$;

This code is used in chunk 12.

## 40. Error Likelihood Ellipse MAPLE Calculation For Example #2

```
> with(stats):
> mesh:=35;
> TPR:=(evalf(osmmts(16/10,2,175/100)[1..2],10));
```

$TPR := 2.097308261, 2.547503694$

```
> RES:=MOGEN(16/10,2,175/100,-12,12,mesh):
```

```
Warning, these names have been redefined: anova, describe, fit,
importdata, random, statevalf, statplots, transform
```

```
> MZ:=J(RES[1],RES[2],mesh):
```

**41.   Graphical Methods For Target Position Report And Error Likelihood Ellipse Display**

The analytical methods demonstrated here relate the graphical depiction of the Error Likelihood Ellipse to the relative position of a given Target Position Report and the most likely position value within the Error Likelihood Ellipse.  These depictions are not meant to list an exhaustive summary of all such graphical methods.  The policy of the PQIC OSMMTS System is to leave such methods to the particular needs of an implementation's circumstances.  The methods found herein are for demonstration purposes only.

```
>  with(plots):
```

**Warning, the name changecoords has been redefined**

```
>  F:=PLOT3D(POLYGONS(seq(seq([
>  [(i-1/2)/mesh,(j-1/2)/mesh,MZ[1][i,j]],
>  [(i+1/2)/mesh,(j-1/2)/mesh,MZ[1][i,j]],
>  [(i+1/2)/mesh,(j+1/2)/mesh,MZ[1][i,j]],
>  [(i-1/2)/mesh,(j+1/2)/mesh,MZ[1][i,j]]],
>  i=MZ[2]..MZ[3]),j=MZ[4]..MZ[5])),
>  COLOR(ZHUE),AXESSTYLE(NORMAL),AXESLABELS("X","Y","Z"),
>  VIEW(TPR[1]-1/2..TPR[1]+1/2,TPR[2]-1/2..TPR[2]+1/2,DEFAULT)):
>  S1:=PLOT3D(POINTS([MZ[6..7]/mesh,0]),COLOR(RGB,0.0,0.0,1.0),SYMBOL(CRO
>  SS,50)):
>  S2:=PLOT3D(POINTS([TPR[1..2],0]),COLOR(RGB,1.0,0.0,0.0),SYMBOL(CROSS,5
>  0)):
>  display({F,S1,S2});
```
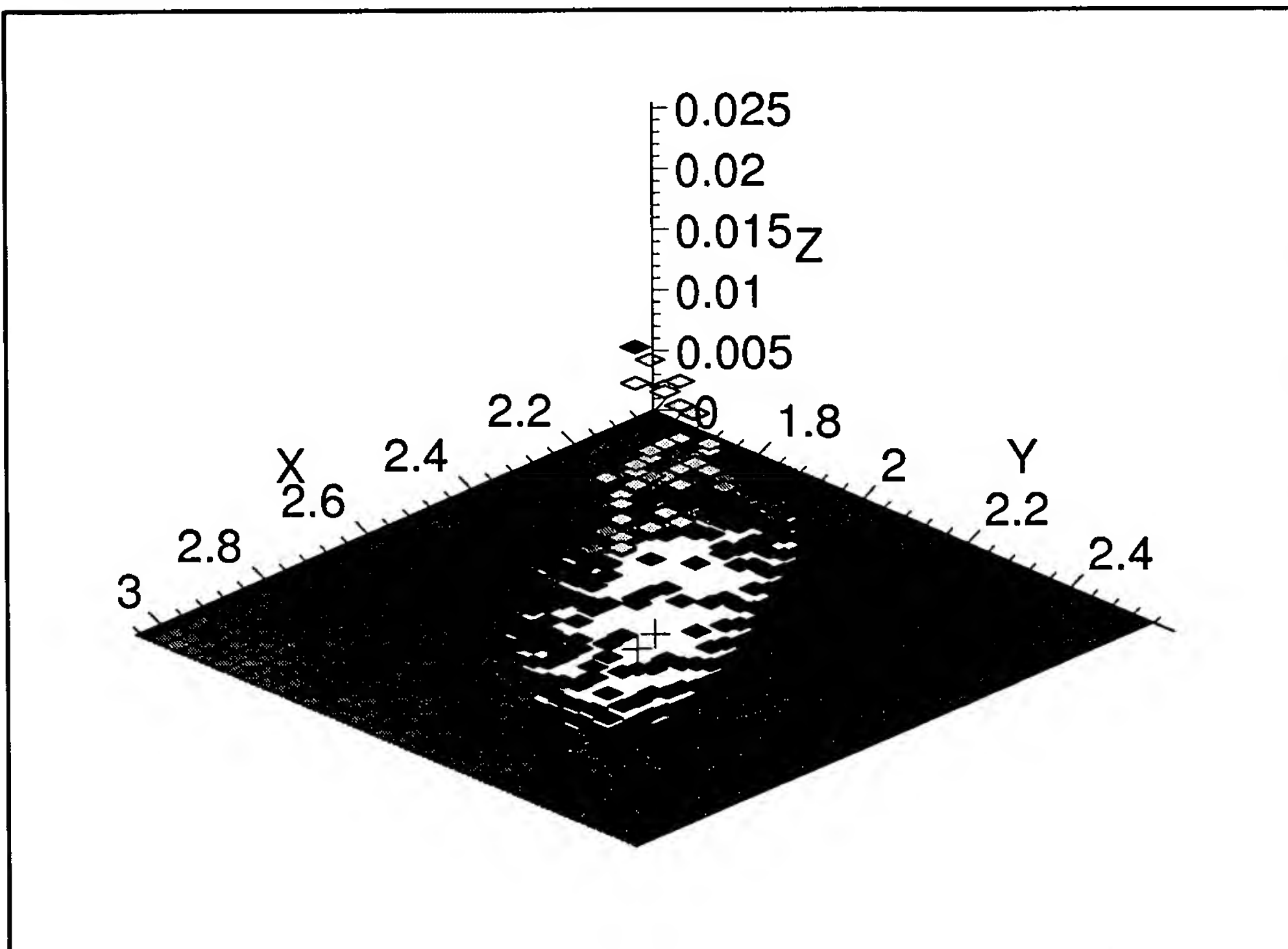


```
>  F:=PLOT3D(POLYGONS(seq(seq([
>  [(i-1/2)/mesh,(j-1/2)/mesh,MZ[1][i-1,j-1]],
>  [(i+1/2)/mesh,(j-1/2)/mesh,MZ[1][i,j-1]],
>  [(i+1/2)/mesh,(j+1/2)/mesh,MZ[1][i,j]],
>  [(i-1/2)/mesh,(j+1/2)/mesh,MZ[1][i-1,j]]],
>  i=MZ[2]+1..MZ[3]),j=MZ[4]+1..MZ[5])),
>  COLOR(ZHUE),AXESSTYLE(NORMAL),AXESLABELS("X","Y","Z"),
>  VIEW(TPR[1]-1/2..TPR[1]+1/2,TPR[2]-1/2..TPR[2]+1/2,DEFAULT)):
>  S:=PLOT3D(POINTS([TPR[1],TPR[2],TPR[3]]),COLOR(RGB,1.0,0.0,0.0),SYMBOL
>  (BOX,50)):
>  display({F,S});
```
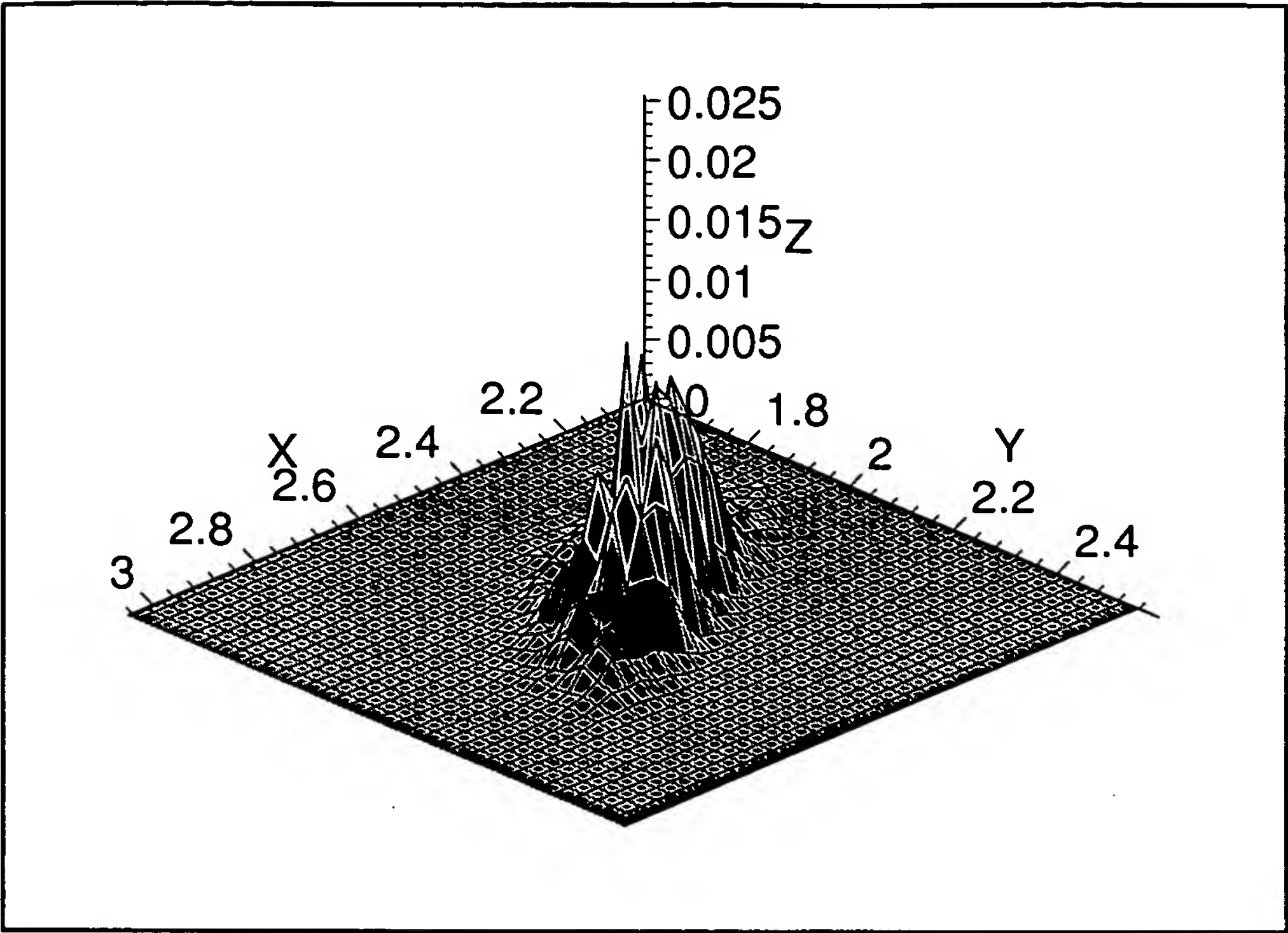
## 42.  The Target Position Report Vector

The *Target Position Report Vector* (TPR) is the quadruple defined by the Calculated Target Position $(x_0, y_0)$, the Maximum Likelihood Target Position $(x'_0, y'_0)$, associated error likelihood ellipse $\left(\frac{x_0^* - x_0}{1 - \frac{m_2}{m_1}}\right)^2 +$ $\left(\frac{y_0^* - y_0}{m_1 - m_2}\right)^2 = R$, and the associated likelihood value for $\frac{(m_1 - m_2)^2}{m_1} R \sim M_3\left(t_1^*, t_2^*, t_3^*\right)$:

$$\mathrm{TPRV} = \left((x_0, y_0), (x'_0, y'_0), \left(\frac{x_0^* - x_0}{1 - \frac{m_2}{m_1}}\right)^2 + \left(\frac{y_0^* - y_0}{m_1 - m_2}\right)^2 = R, \frac{(m_1 - m_2)^2}{m_1} R \sim M_3\left(t_1^*, t_2^*, t_3^*\right)\right)$$

The value of $R$ is chosen so that $\frac{(m_1 - m_2)^2}{m_1} R$ is at the $100(1 - \alpha)\%$ percentile of the $M_3\left(t_1^*, t_2^*, t_3^*\right)$ distribution.

## 43. Containment Policies

## 44. Definitions and Remarks

**Definition 14** *A TPR is said to be* accurate *if the actual position of the target is inside the associated ELE. The value of R determines the likelihood of this event.*

**Definition 15** *Any calculation algorithm used to produce a set of numerical values intermediate and inferior to the TPR is called an* analytical step.

**Definition 16** *An analytical step is called a* mitigation *if it is taken before the time data $\{t_1, t_2, \ldots, t_k, \ldots\}$ are collected.*

**Definition 17** *An analytical step is called a* optimization *if it occurs after the time data $\{t_1, t_2, \ldots, t_k, \ldots\}$ are collected.*

**Remark 18** *The purpose of mitigation steps is to reduce the error variance $\sigma$.*

**Remark 19** *The purpose of optimization steps is to increase the likelihood of an accurate TPR.*

**Definition 20** *An irregularly occurring, non-analytical step taken at any time to accomplish the same goals as mitigation and optimization is called* ad-hoc.

**Definition 21** *The collection of ad-hoc, mitigation, or optimization steps taken in an implementation of the OSMMTS is called the system's* containment policies, *and referred to individually as a system* containment policy.

## 45. The Demerit System

The OSMMTS *Demerit System* is an ad-hoc containment policy that acts simultaneously as a mitigation and an optimization. Under this system, the three SDU's chosen to calculate the TPR are those three that are most likely to produce the "best" TPR based on past performance (thereby making it an optimization step), by way of reducing the variability of the utilized data (thereby making it a mitigation step).

Suppose there are $n$-many SDU's, however, only $k \leq n$ many receive a signal within the reception window. There are $\binom{n}{k}$-many combinations of SDU's, and $\binom{k}{3}$-many combinations of the $k$-many that receive the signal taken three at a time. Each SDU has three values associated with it at the beginning of each processing cycle, namely its non-negative *Demerit Count*, its positive *History Total*, and its possibly null Boolean *Confirmation Value*. At the beginning of all processing, the demerit count for each SDU will be zero, the history total with be one, and the confirmation value will be null. The confirmation value at the beginning of the processing cycle is determined by its observed value during the confirmation cycle. At the end of a processing cycle, the demerit count and history total are determined by the steps below, and the confirmation value is set back to null.

For each processing cycle, and for each of the $\binom{k}{3}$-many combinations, the following steps determine the end-of-processing-cycle demerit counts and history totals.

1. Set the likelihood value $\lambda$.

2. Eliminate those $\tau_1$-many combinations that are collinear.

3. Eliminate those $\tau_2$-many combinations that do not all have positive history totals and TRUE confirmation values.

    The SDU's involved in the $(\tau_1 + \tau_2)$-many combinations eliminated in Steps 2-3 are called *deficient* for the current processing cycle. This designation is removed at the beginning of a new processing cycle.

4. Among the remaining, i.e., *qualifying* combinations, choose the combination of three SDU that collectively have the minimal sum of demerits.

5. In case of a tie in Step 4, use the combination with the largest history sum. In case of a further tie, choose the combination with the smallest individual demerit count. In case of a last tie, randomly choose uniformly among the finalists.

    The combination so chosen is called the *calculating* combination, and the SDU's involved are called the *elected* SDU's. Increment the history total by 1 for each elected SDU.

6. Subtract two demerits from the count for each elected SDU. Recall the demerit count for an SDU cannot become negative.

7. Calculate the TPR using the calculating combination.

8. Calculate the $\lambda$-ELE for the calculating combination.

9. Calculate the $\left(\binom{k}{3} - (\tau_1 + \tau_2)\right)$-many TPR for all other qualifying combinations. Each of these TPR is called an Alternate Position Report (APR).

10. For each APR calculated in Step 7, if the APR falls outside the $\lambda$-ELE, then add one demerit to the count for each SDU involved in the APR.

11. For each APR calculated in Step 7, if the APR falls inside or on the $\lambda$-ELE, then subtract one demerit to the count for each SDU involved in the APR. Recall the demerit count for an SDU cannot become negative.

12. Add one demerit for each SDU that does not report a positive confirmation.

13. When the demerit count for an SDU exceeds the *Warning Threshold*, send an alert to report a frequently deficient SDU.

14. When the demerit count for an SDU exceeds the *Terminal Threshold*, shut down communication with the SDU and do not consider it further (by setting its history total to zero) until explicitly reset. Also send an alert to report a failed SDU.

15. These steps are in addition to the disabling of an SDU if proper query responses are not confirmed during the receive phase.

## 46. Mitigations For Reflections

Given a TPR $(x_0, y_0)$ and multiple detection times at an SDU, say $\{t_1, t_2, t_3, \ldots\}$, where $t_1$ is used to generate $(x_0, y_0)$, the question of mitigating the error in the TPR due to reflections[13] of the "true" signal may be stated as follows: What is the earliest time a reflection of the "true" signal could be detected at an SDU, and what is the latest such time. Any signal detected between these two extremes would be considered a reflection, and therefore not used in the calculation of a TPR. Any signal detected "too early" or "too late" would indicate the detection time data is corrupt and should not be used to calculate the TPR.

**Axiom 22** *If the signal detection times recorded during the Receive Phase are inconsistent with a calculated TPR, the TPR should not be used.*

## 47. Definitions
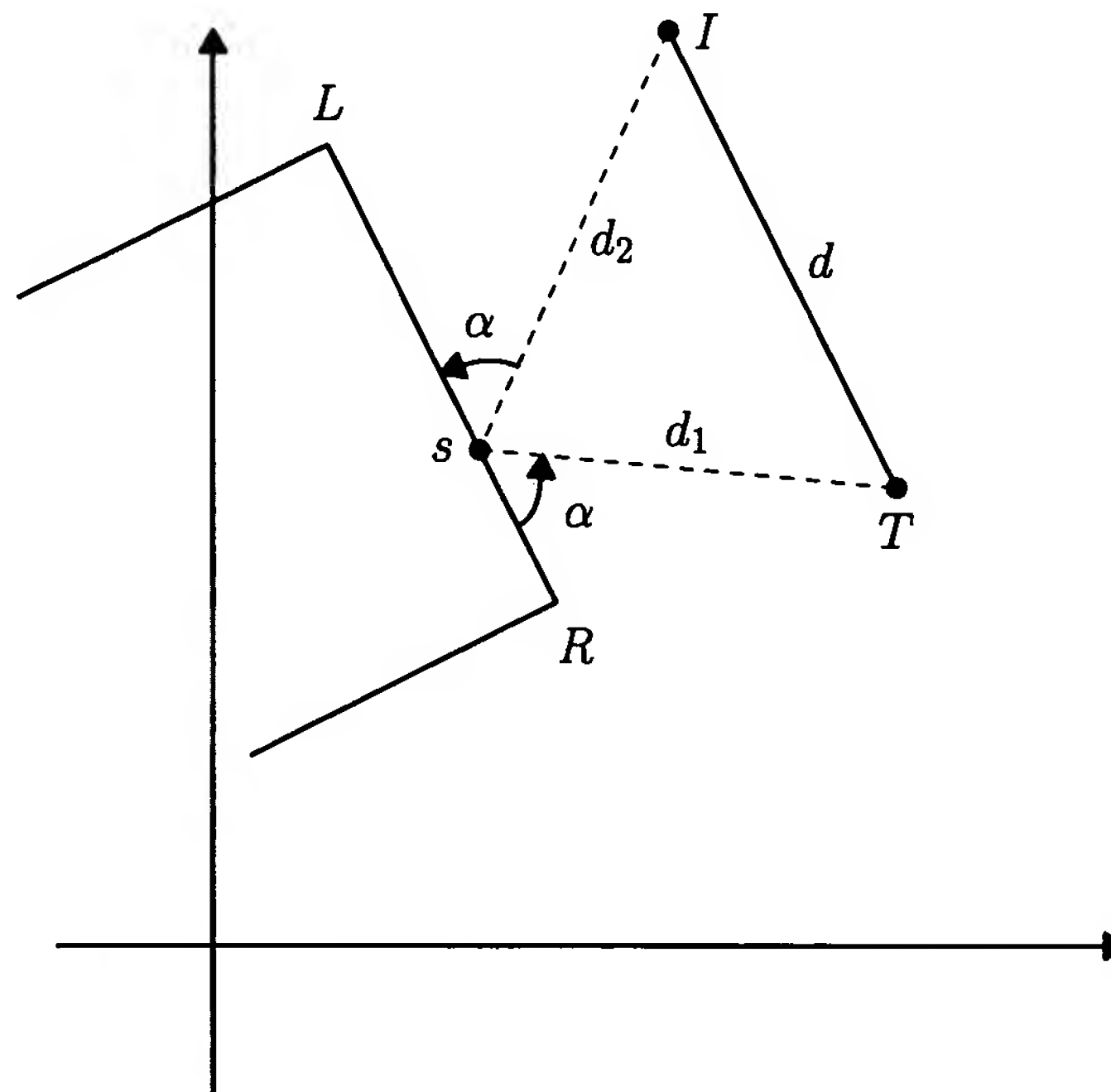


**Figure 10**: Generic Straight Line Obstruction

Consider a generic straight line obstruction as depicted in Figure 10. Even though the following development requires a straight line obstruction, a more generalized parameterized formulation may be substituted for 14, such as a Bézier curve or trigonometric function.

As $s$ goes from 0 to 1, point $(x_s, y_s)$ goes from $L = (x_l, y_l)$ to $R = (x_r, y_r)$, so that

$$
\begin{aligned}
(x_s, y_s) &= s(x_r, y_r) + (1 - s)(x_l, y_l) \\
&= (x_l + s(x_r - x_l), y_l + s(y_r - y_l))
\end{aligned}
\tag{14}
$$

Then given SDU position $(x_1, y_1)$, TPR position $(x_0, y_0)$, and obstruction endpoints $L$ and $R$, we wish to find

$$
d_{\min} = \min_{0 \le s \le 1} \{d_1 + d_2\}
$$

and

$$
d_{\max} = \max_{0 \le s \le 1} \{d_1 + d_2\}
$$

subject to

$$
d^2 = d_1^2 + d_2^2 - 2d_1 d_2 \cos(\pi - 2\alpha)
$$

---

[13] The term "reflections" also includes signal behavior characterized by "echos," "secondary," and "phantom."

since the angle of incidence $\alpha$ would be consistent "inbound" to, and "outbound" from, the obstruction. Define the *Window for Reflections Between* **I** *and* **T** to be

$$\frac{d_{\min}}{c} \leq t \leq \frac{d_{\max}}{c} \tag{15}$$

Note that at point $(x_s, y_s)$ we have

$$\alpha = \arccos\left(\frac{(x_0 - x_s)*(x_r - x_s) + (y_0 - y_s)*(y_r - y_s)}{\sqrt{(x_0 - x_s)^2 + (y_0 - y_s)^2}\sqrt{(x_r - x_s)^2 + (y_r - y_s)^2}}\right)$$

so that

$$
\begin{aligned}
\cos(\pi - 2\alpha) &= -\cos 2\alpha \\
&= 1 - 2\cos^2\alpha \\
&= 1 - \frac{2((x_0-x_s)*(x_r-x_s)+(y_0-y_s)*(y_r-y_s))^2}{((x_0-x_s)^2+(y_0-y_s)^2)((x_r-x_s)^2+(y_r-y_s)^2)} \\
&= 1 - 2\frac{\left(\begin{array}{c}((x_0-x_l)-s(x_r-x_l))*((1-s)(x_r-x_l)) \\ +((y_0-y_l)-s(y_r-y_l))*((1-s)(y_r-y_l))\end{array}\right)^2}{\left(\begin{array}{c}((x_0-x_l)-s(x_r-x_l))^2 \\ +((y_0-y_l)-s(y_r-y_l))^2\end{array}\right)\left(\begin{array}{c}((1-s)(x_r-x_l))^2 \\ +((1-s)(y_r-y_l))^2\end{array}\right)} \\
&= \gamma(s)
\end{aligned}
$$

and

$$d_1(s) = \sqrt{((x_1 - x_l) - s(x_r - x_l))^2 + ((y_1 - y_l) - s(y_r - y_l))^2}$$

and

$$d_2(s) = \sqrt{((x_0 - x_l) - s(x_r - x_l))^2 + ((y_0 - y_l) - s(y_r - y_l))^2}$$

which means $d^2(s)$ may be viewed as a function of $s$.

## 48.  Policies

The *Mitigation for Reflections Method* calculates

$$\frac{d_{\min}}{c} = \frac{1}{c}\min_{0 \leq s \leq 1}\{d_1 + d_2\} \quad \text{and} \quad \frac{d_{\max}}{c} = \frac{1}{c}\min_{0 \leq s \leq 1}\{d_1 + d_2\}$$

subject to

$$d^2(s) = d_1^2(s) + d_2^2(s) - 2d_1(s)d_2(s)\gamma(s)$$

for each combination of SDU position, possible target position, and $L$ and $R$ positions for a straight line obstruction that would potentially corrupt the TPR through the detection of reflections (see again Figure 10).

Given a TPR $(x_0, y_0)$ and multiple detection times $\{t_1, t_2, t_3, \ldots\}$ at SDU **I**, where $t_1$ is used to calculate the TPR, if

$$\frac{d_{\min}}{c} \leq t_i \leq \frac{d_{\max}}{c}$$

for *all* $i = 2, 3, \ldots$, then the TPR is accepted, i.e., there is no need for a mitigation for reflections
If, however,

$$t_i < \frac{d_{\min}}{c} \quad \text{or} \quad \frac{d_{\max}}{c} < t_i$$

for *any* $i = 2, 3, \ldots$, then the TPR is rejected, and a new TPR is calculated using the next highest priority qualifying combination of SDU's. This cycle continues until a TPR is accepted or the qualifying combination of SDU's is depleted.

## 49.  MAPLE Implementation Code

**50.**  ⟨Mitigations For Reflections 50⟩ ≡
   ⟨Mitigation Basis Algorithm 51⟩
   ⟨Mitigation Extrema Algorithm 52⟩
This code is used in chunk 12.

**51.** ⟨Mitigation Basis Algorithm 51⟩ ≡

$mitig: = proc(xx\_0, yy\_0)$

$local\ s, x\_s, y\_s, d, d\_1, d\_2, g, s0, x0\_s, y0\_s;$

$global\ x\_l, y\_l, x\_r, y\_r, x\_1, y\_1;$

$options\ 'Copyright\ 2003\ PQI\ Consulting\ All\ Rights\ Reserved';$

$description\ "OSMMTS_\sqcup Mitigation_\sqcup Algorithm";$

$\textbf{if}\ (xx\_0, yy\_0) = (x\_1, y\_1)\ \textbf{then}\ \textbf{return}\ -1, -1, -1, -1\ \textbf{end if};$

$x\_s: = x\_l + s * (x\_r - x\_l);$

$y\_s: = y\_l + s * (y\_r - y\_l);$

$d\_1: = sqrt((((x\_1 - x\_l) - s * (x\_r - x\_l)) \oplus 2) + (((y\_1 - y\_l) - s * (y\_r - y\_l)) \oplus 2));$

$d\_2: = sqrt((((xx\_0 - x\_l) - s * (x\_r - x\_l)) \oplus 2) + (((yy\_0 - y\_l) - s * (y\_r - y\_l)) \oplus 2));$

$d: = sqrt((x\_1 - xx\_0) \oplus 2 + (y\_1 - yy\_0) \oplus 2);$

$g: = simplify\ (\ 1 - 2 * (((((xx\_0 - x\_l) - s * (x\_r - x\_l)) * ((1 - s) * (x\_r - x\_l))$
$+ ((yy\_0 - y\_l) - s * (y\_r - y\_l)) * ((1 - s) * (y\_r - y\_l))) \oplus 2) / ((((xx\_0 - x\_l) - s * (x\_r - x\_l)) \oplus 2 + ((yy\_0 - y\_l)$
$- s * (y\_r - y\_l)) \oplus 2) * (((1 - s) * (x\_r - x\_l)) \oplus 2 + ((1 - s) * (y\_r - y\_l)) \oplus 2))\ )\ );$

$s0: = fsolve(d\_1 \oplus 2 + d\_2 \oplus 2 - 2 * d\_1 * d\_2 * g = d, s, s = 0..1);$

$\textbf{if}\ substring\ (convert(s0, string), 1..3) <> "fso"\ \textbf{then}$

$\textbf{return}\ s0, subs(s = s0, x\_s), subs(s = s0, y\_s), subs(s = s0, d\_1 + d\_2);$

$\textbf{else}\ \textbf{return}\ -1, minimize(g, s = 0..1), maximize(g, s = 0..1), -1;$

$\textbf{end if};$

This code is used in chunk 50.

**52.** ⟨Mitigation Extrema Algorithm 52⟩ ≡

$xmit: = proc()$

$local\ res, minres, maxres, u, v;$

$global\ x\_0, y\_0;$

$options\ 'Copyright\ 2003\ PQI\ Consulting\ All\ Rights\ Reserved';$

$description\ "OSMMTS_\sqcup Mitigation_\sqcup Extrema_\sqcup Calculation";$

$minres: = mitig(x\_0, y\_0)[4];$

$maxres: = minres;$

$\textbf{for}\ u\ \textbf{from}\ -30\ \textbf{to}\ 30\ \textbf{do}\ \textbf{for}\ v\ \textbf{from}\ -30\ \textbf{to}\ 30\ \textbf{do}\ res: = mitig(x\_0 + u/10, y\_0 + v/10)[4];$

$\textbf{if}\ res <> -1\ \textbf{then}\ \textbf{if}\ res < minres\ \textbf{then}\ minres: = res\ \textbf{end if};$

$\textbf{if}\ res > maxres\ \textbf{then}\ maxres: = res\ \textbf{end if};$

$\textbf{end if};$

$\textbf{end do};$

$\textbf{end do};$

$\textbf{return}\ minres, maxres;$

$\textbf{end}\ proc;$

This code is used in chunk 50.

**53. Sample Calculations And Results**

```
>    (x_l,y_l):=(1,5);  (x_r,y_r):=(2,2);  (x_0,y_0):=(4,3);
>    (x_1,y_1):=(3,6);
```

$$x\_l, y\_l := 1, 5$$
$$x\_r, y\_r := 2, 2$$
$$x\_0, y\_0 := 4, 3$$
$$x\_1, y\_1 := 3, 6$$

```
>    mitig(x_0-1,y_0+3);
```

$$-1, -1, -1, -1$$

```
>    xmit();
```

$$3.493116110, 9.112507778$$

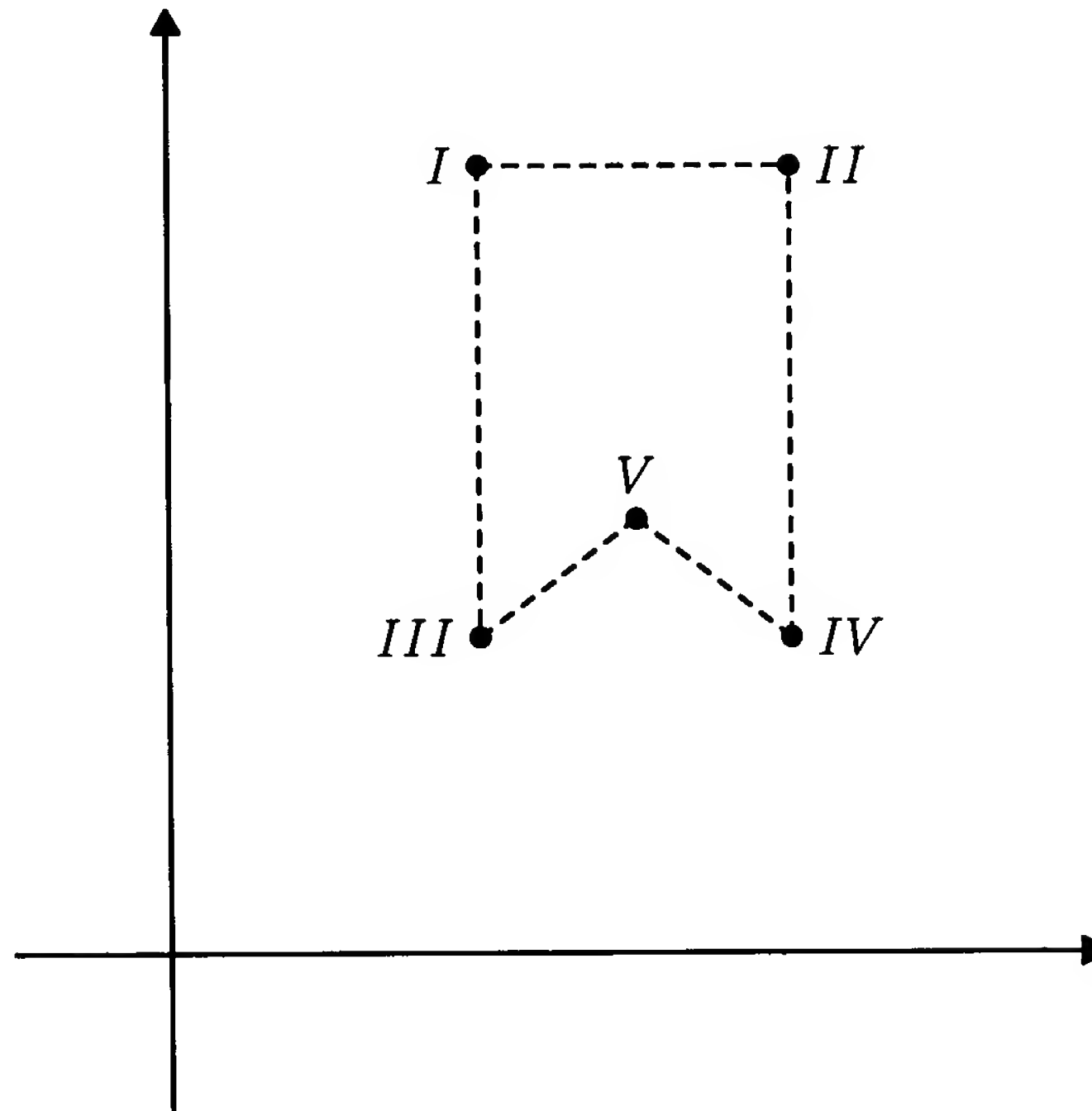## 54.  Optimal Placement Of Detection Units

Figure 11: Poor Placement Of SDU Units

The optimal placement of SDU's depends on the OSMMTS goals for the implementation under consideration. In general, a maximum likelihood approach to SDU placement works well for many applications. Its goals address the probability of an accurate TPR calculation with the first qualifying set of SDU's within a particular SDU arrangement. The probability of such placement is maximized among all convex polygons that encloses all possible target locations, or those target locations that are of interest in the implementation, or where the addition of any more SDU's would not enclose a significantly larger area than is covered by smaller set of SDU's. Such an approach uses the following rules for calculating a set of exact SDU positions for optimal use.

### Optimized SDU Placement Rules

1. SDU's should be placed where the connected units form a convex hull (see Figure 12 for an example of a convex hull, and Figure 11 for an example of a non-convex hull).

2. The convex hull should cover as much of the possible target locations as possible.

3. SDU's are placed so that no three units are collinear.

4. SDU's are placed so that no two units have the same $x$-value or $y$-value, as viewed in a coordinate grid.

5. SDU's should be placed to maximize the likelihood that $d_i > |f_i| c$, for all $i$.

No consideration is made in these rules for obstructions, or other exceptions in a clear area of target detection. Under the OSMMTS, such considerations are handled by mitigations. Furthermore, if the set of possible target locations, or those target locations that are of interest in the implementation, contain "gaps," regardless of size and shape, such gaps shall not be considered in the optimal placement of detection units, since their consideration may be completely addressed by the methods found in the containment policies. Specialized implementation of the OSMMTS may introduce special SDU placement considerations, which will be documented under separate cover in the particular implementation documentation.

Figure 13 shows an arrangement of five SDU's that satisfy the guidelines embedded in the placement rules. Since the position of the origin is arbitrary, the coordinate center of the region may be translated to any local coordinate system within any particular OSMMTS implementation.
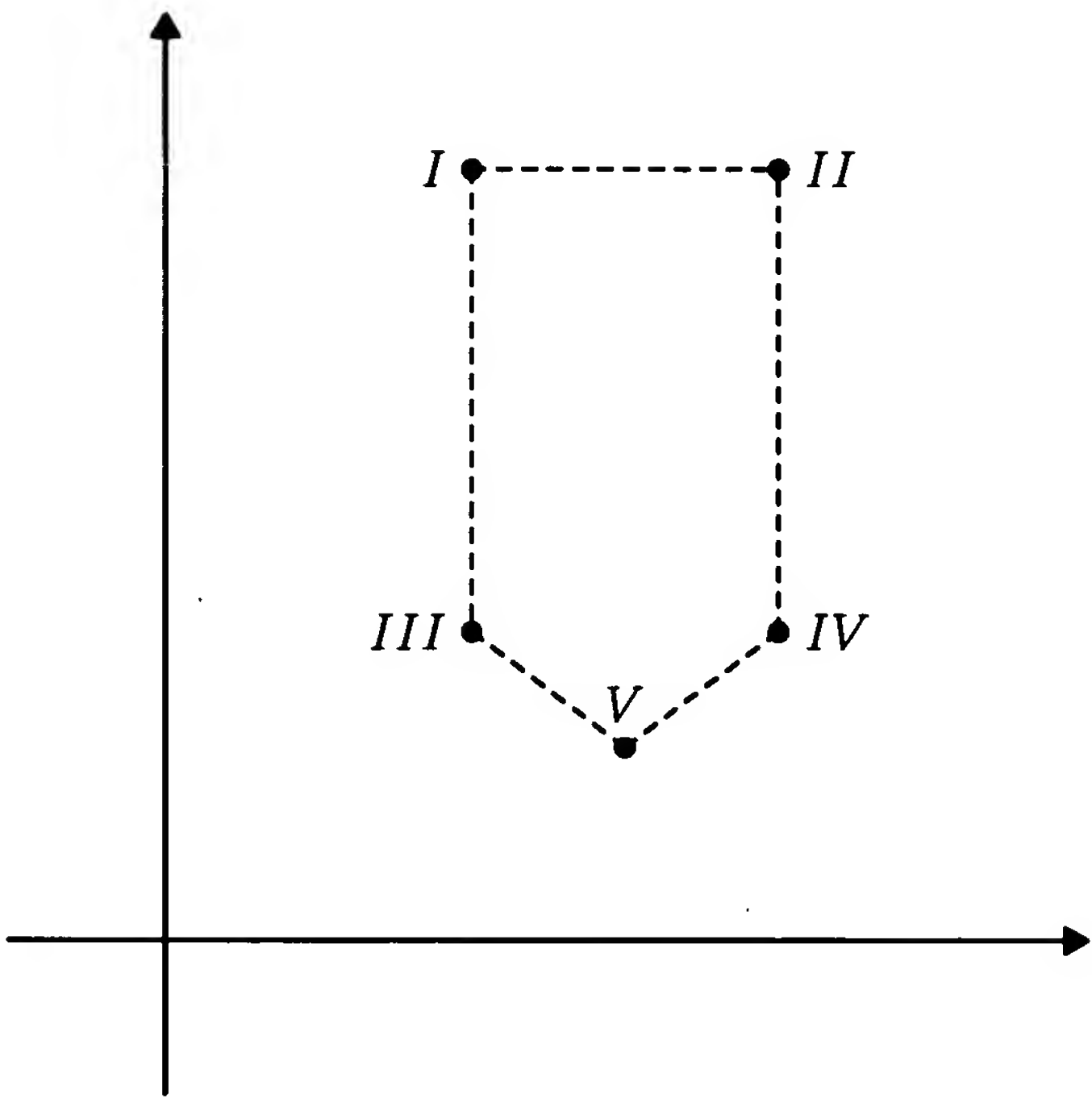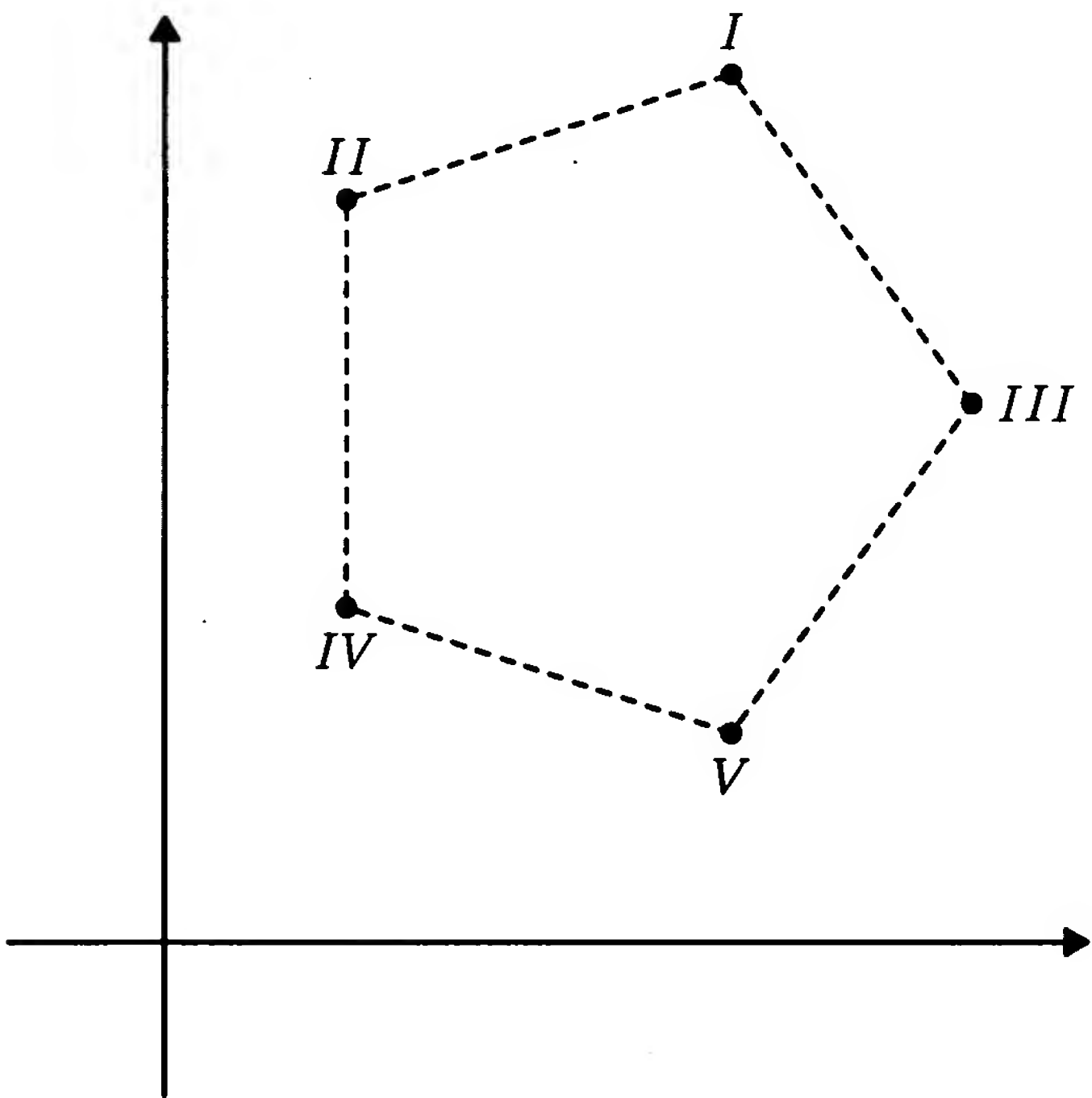
**Figure 12**: Better Placement Of SDU Units

**Figure 13**: Best Placement Of SDU Units

## 55. The Convex Hull Principle

Since the area of any segmented convex hull is maximized when the distances between the vertices of the segments are equal[14], the "best" convex hull, i.e., that which follows the placement rules, is based on a circle

---

[14] Since all convex polyhedra may be subdivided into triangles, it suffices to prove the following claim to make this assertion.

**Claim 23** *Given a fixed perimeter length, the triangle with maximum area is equilateral.*

*Proof.* Let $a > 0, b > 0$, and $c > 0$ be the length of the sides of a triangle. Then the area of such a triangle is given by

$$A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

where $s = \frac{1}{2}\,(a+b+c)$. It remains to show that $A$ achieves its maximum when $a = b = c$, given a fixed value of $a + b + c$. Consider the Lagrange Multiplier Function $K$:

$$K\,(a,b,c) = A - \lambda\,(a+b+c)$$

Then

$$\frac{\partial K}{\partial a} = \frac{\partial A}{\partial a} - \lambda = \frac{B_1}{4A} - \lambda$$

where

$$B_1 = \left\{ \begin{array}{l} (-a+b+c)\,(a-b+c)\,(a+b-c) \\ -(a+b+c)\,(a-b+c)\,(a+b-c) \\ +(a+b+c)\,(-a+b+c)\,(a+b-c) \\ +(a+b+c)\,(-a+b+c)\,(a-b+c) \end{array} \right\}$$

Similarly, we have

$$\frac{\partial K}{\partial b} = \frac{\partial A}{\partial b} - \lambda = \frac{B_2}{4A} - \lambda$$

where

$$B_2 = \left\{ \begin{array}{l} (-a+b+c)\,(a-b+c)\,(a+b-c) \\ +(a+b+c)\,(a-b+c)\,(a+b-c) \\ -(a+b+c)\,(-a+b+c)\,(a+b-c) \\ +(a+b+c)\,(-a+b+c)\,(a-b+c) \end{array} \right\}$$

Setting

$$\frac{\partial K}{\partial a} = 0 = \frac{\partial K}{\partial b}$$

we have

$$\lambda = \frac{B_1}{4A} = \frac{B_2}{4A}$$

or

$$\left\{ \begin{array}{l} -(a+b+c)\,(a-b+c)\,(a+b-c) \\ +(a+b+c)\,(-a+b+c)\,(a+b-c) \end{array} \right\} = \left\{ \begin{array}{l} +(a+b+c)\,(a-b+c)\,(a+b-c) \\ -(a+b+c)\,(-a+b+c)\,(a+b-c) \end{array} \right\}$$

or

$$a - b + c = -a + b + c$$

which means

$$a = b$$

Furthermore,

$$\frac{\partial K}{\partial a} = 0 = \frac{\partial K}{\partial c}$$

gives

$$\left\{ \begin{array}{l} -(a+b+c)\,(a-b+c)\,(a+b-c) \\ +(a+b+c)\,(-a+b+c)\,(a-b+c) \end{array} \right\} = \left\{ \begin{array}{l} +(a+b+c)\,(a-b+c)\,(a+b-c) \\ -(a+b+c)\,(-a+b+c)\,(a-b+c) \end{array} \right\}$$

or

$$a + b - c = -a + b + c$$

which means

$$a = c$$

In fact,

$$\frac{\partial^2 K}{\partial a^2} = \frac{4AB_{11} - \frac{B_1^2}{A}}{16A^2}$$

where

$$B_{11} = \left\{ \begin{array}{c} -2\,(a-b+c)\,(a+b-c) \\ +4a\,(-a+b+c) \\ +2\,(a+b+c)\,(-3a+b+c) \end{array} \right\}$$

Then $a = b = c$ gives

$$B_{11}\big|_{a=b=c} = \left\{ \begin{array}{c} -2a^2 \\ +4a^2 \\ -6a^2 \end{array} \right\} = -4a^2$$

and

$$A\big|_{a=b=c} = \sqrt{\left(\frac{3}{2}a\right)\left(\frac{3}{2}a-a\right)\left(\frac{3}{2}a-a\right)\left(\frac{3}{2}a-a\right)} = \frac{1}{4}\sqrt{3a^4} = \frac{\sqrt{3}}{4}a^2$$

(which would be an "infinite-many sided polygon"), and the $n$-many SDU's would be at positions

$$re^{\frac{2k\pi}{n}i} = r\left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)$$

for $k = 0, 1, 2, \ldots, n-1$, and where $r$ is determined by the effective range of the possible target positions, relative to an arbitrarily chosen "origin" for location purposes only. Note how such an arrangement automatically satisfies Placement Rules #1 and #3, and Placement Rule #5 is violated only when a target comes between the SDU's on the "edges" of the containing circle. This consideration may be addressed through the choice for $r$, which also covers Placement Rule #2.

The value of $r$ is called the *Radius of Coverage*, and the placement of the units so described is called the *SDU Arrangement*.

For example, for $n = 8$, the SDU's would be at positions $e^{\frac{k\pi}{4}i}$, for $k = 0, 1, 2, \ldots, 7$. This translates to

$$r\left(\cos\eta + i\sin\eta\right)$$

where

$$\eta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$$

(see Figure 13).

Note how, for $n = 8$, there is a problem with having SDU's at both

$$r\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \text{ and } r\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

since $\sin\frac{\pi}{4} = \sin\frac{3\pi}{4}$, which violates Placement Rule #4. However, choosing any *prime* $n > 4$ satisfies[15] Placement Rule #4.

The optimal placement of SDU's may now be characterized analytically in terms of the "point of diminishing returns."

**Condition 24** *Given a coverage function $A(n, r)$ that describes the area covered by the $n$-many SDU arrangement with radius of coverage $r$, find*

$$n_*(r \mid \xi) = \min_n\left\{\frac{\partial A}{\partial n}(n, r) < \xi\right\}$$

Then given $r$, $n_*(r)$ is the minimum number of SDU's necessary to satisfy the placement rules, and where the coverage area instantaneous rate of increase first drops below $\xi$ units of area. By choosing a sufficiently robust value for $\xi$, the optimal number $n_*$ may be explicitly calculated.

## 56. Formulation of $A(n, r)$

Since the SDU arrangement is completely symmetric for any two SDU's and the center of the containing circle, consider the sector spanned by two SDU's $A$ and $B$, and the center of the containing circle $O$ (see Figure 14). Note that the position and orientation of this figure is arbitrary, and the results of this section do not depend on the particulars of such aspects.

**Lemma 25** *The area of $\triangle AOB$ is $\frac{1}{2}r^2\sin\left(\pi\left(1 - \frac{2}{n}\right)\right)$.*

---

and

$$\left.\frac{\partial^2 K}{\partial a^2}\right|_{a=b=c} = \frac{\sqrt{3}a^2\left(-4a^2\right) - \frac{\left(4a^3\right)^2}{\frac{\sqrt{3}}{4}a^2}}{16\left(\frac{\sqrt{3}}{4}a^2\right)^2} = \frac{\left(-4\sqrt{3} - \frac{64}{\sqrt{3}}\right)a^4}{3a^4} = -\frac{4}{3}\left(\sqrt{3} + \frac{16}{\sqrt{3}}\right) < 0$$

regardless of the value of $a$. Similar calculations confirm $\left.\frac{\partial^2 K}{\partial b^2}\right|_{a=b=c} < 0$ and $\left.\frac{\partial^2 K}{\partial c^2}\right|_{a=b=c} < 0$ regardless of the value of $b$ or $c$. ∎

[15] If $n$ is prime, then

$$w_1 \neq w_2 \implies \cos\frac{2w_1\pi}{n} \neq \cos\frac{2w_2\pi}{n}$$

and

$$w_1 \neq w_2 \implies \sin\frac{2w_1\pi}{n} \neq \sin\frac{2w_2\pi}{n}$$

since $n$ would not be divisible by 2, or by any $w_i$. Hence, no two SDU's would have the same $x$-value or same $y$-value.
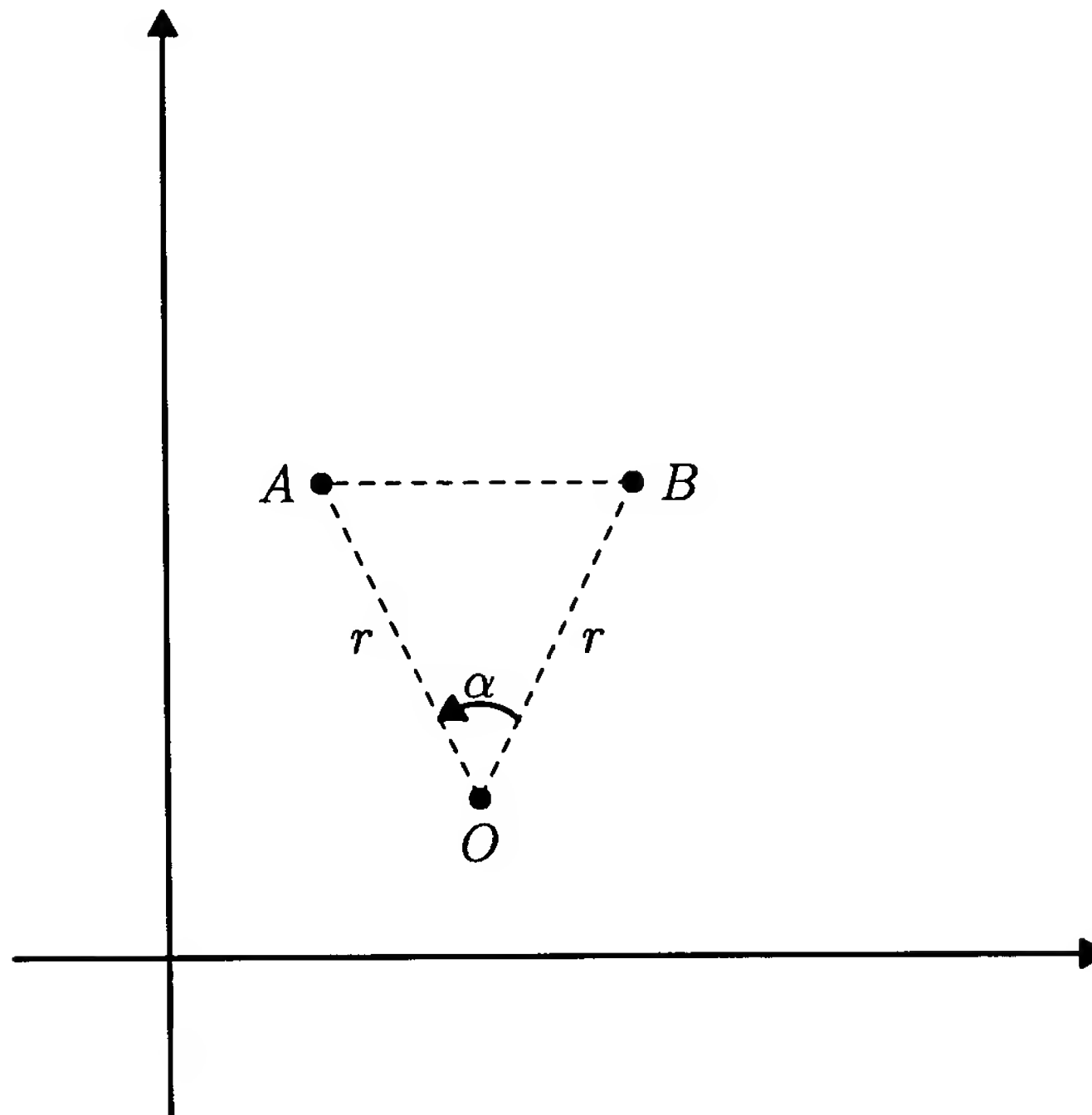
**Figure 14:** Optimal Placement Sector

*Proof.* In Figure 15, $\triangle COB$ is a right triangle with height $\overline{OC}$ and base $\overline{CB}$. Since $\overline{OB} = \overline{OA}$, then $\beta = \frac{\pi - \alpha}{2}$, corresponding to $\angle OAC$ and $\angle OBC$.

Then

$$\overline{OC} = r \sin \beta \text{ and } \overline{CB} = r \cos \beta$$

which means

$$\text{Area Of } \triangle COB = \frac{1}{2} r^2 \sin \beta \cos \beta$$

Since $\triangle COB$ is similar to $\triangle COA$, then

$$\text{Area Of } \triangle AOB = r^2 \sin \beta \cos \beta = \frac{1}{2} r^2 \sin 2\beta$$

In the SDU arrangement, we have $\alpha = \frac{2\pi}{n}$, for $n > 4$. Hence, the area covered by the convex hull of SDU's is incrementally given by $\frac{1}{2} r^2 \sin 2\beta$, where

$$\beta = \frac{\pi - \left( \frac{2\pi}{n} \right)}{2} = \frac{n\pi - 2\pi}{2n} = \pi \left( \frac{n-2}{2n} \right) = \pi \left( \frac{1}{2} - \frac{1}{n} \right)$$

Hence, for an $n$-many SDU arrangement, we have

$$\text{Area Of } \triangle AOB = \frac{1}{2} r^2 \sin 2\beta = \frac{1}{2} r^2 \sin \left( \pi \left( 1 - \frac{2}{n} \right) \right)$$

■

**Claim 26** $A(n,r) = \frac{n}{2} r^2 \sin \left( \pi \left( 1 - \frac{2}{n} \right) \right)$

*Proof.* The coverage region is comprised on $n$-many exactly sized sectors of the type found in Figure 14. The result follows immediately by Lemma 25. ■

For a fixed $r$, as $n \to \infty$, we have

$$\pi \left( 1 - \frac{2}{n} \right) \to \pi$$

and

$$\sin \left( \pi \left( 1 - \frac{2}{n} \right) \right) \to 0$$
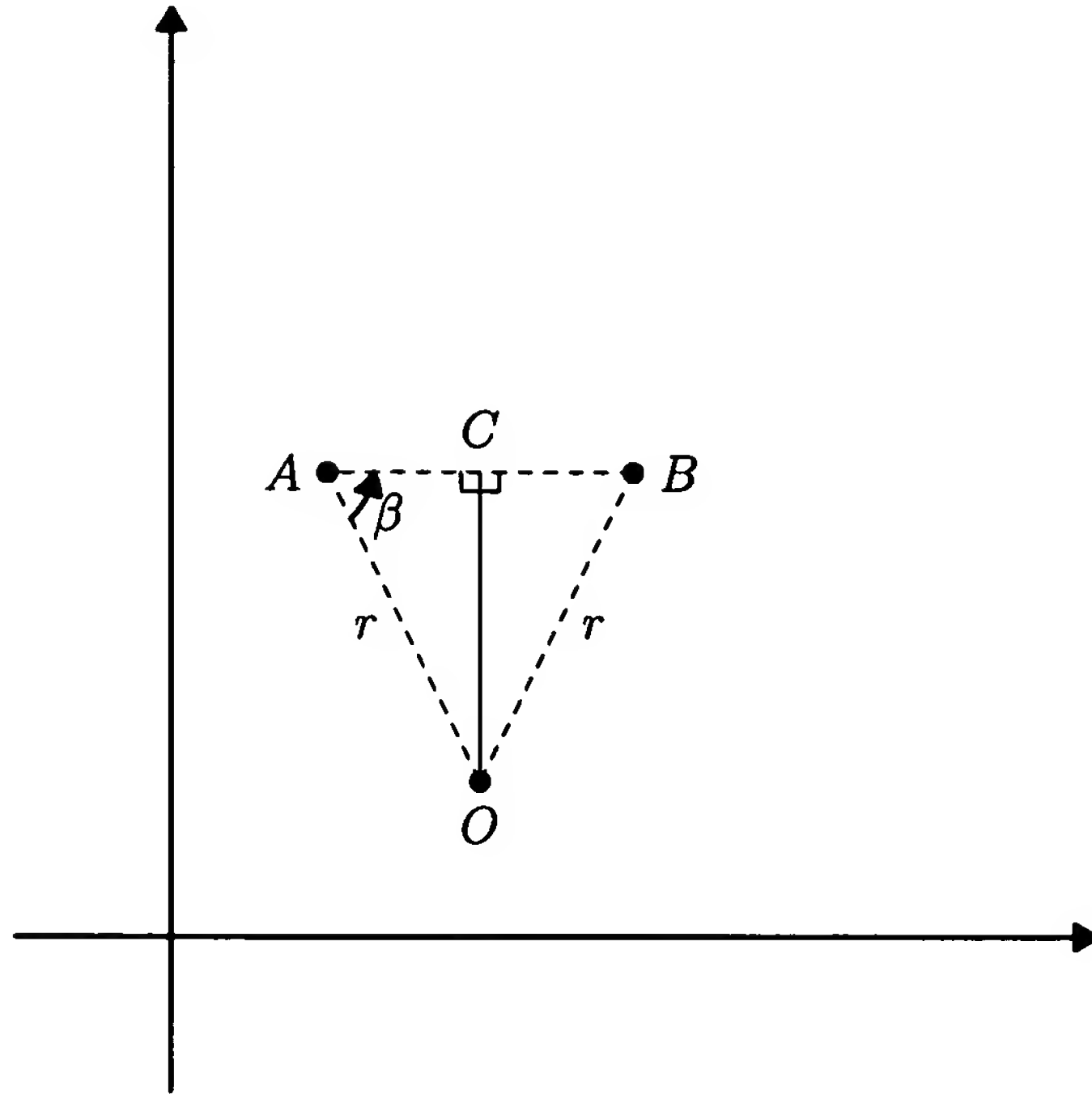
**Figure 15**: Sector Division

In fact,

$$
\begin{aligned}
\lim_{n\to\infty} \tfrac{n}{2} r^2 \sin\left(\pi\left(1-\tfrac{2}{n}\right)\right)
&= r^2 \lim_{n\to\infty} \frac{\sin\left(\pi\left(1-\tfrac{2}{n}\right)\right)}{\tfrac{2}{n}} \\
&= r^2 \lim_{n\to\infty} \frac{\tfrac{2\pi}{n^2}\cos\left(\pi\left(1-\tfrac{2}{n}\right)\right)}{-\tfrac{2}{n^2}} \\
&= -\pi r^2 \lim_{n\to\infty} \cos\left(\pi\left(1-\tfrac{2}{n}\right)\right) \\
&= -\pi r^2 \left(-1\right) \\
&= \pi r^2
\end{aligned}
$$

which is the area of a perfect circle of radius $r$.

Now by Claim 26, we have

$$
\frac{\partial A}{\partial n}(n,r) = r^2\left(\frac{1}{2}\sin\left(\pi\left(1-\frac{2}{n}\right)\right) + \frac{\pi}{n}\cos\left(\pi\left(1-\frac{2}{n}\right)\right)\right)
$$

the graph of which is contained in Figure 16. This may be used to calculate $n_*$ for a given $\xi$.

Finally, solving $A(n,r)$ for $r$ gives

$$
r = \sqrt{\frac{2}{n}A\csc\left(\pi\left(1-\frac{2}{n}\right)\right)}
$$

and if the approximate coverage area $A$ and the approximate radius of coverage $r$ are given, then define

$$
n_{**} = \left\lfloor \frac{2}{r^2}A\csc\mu \right\rfloor + 1
$$
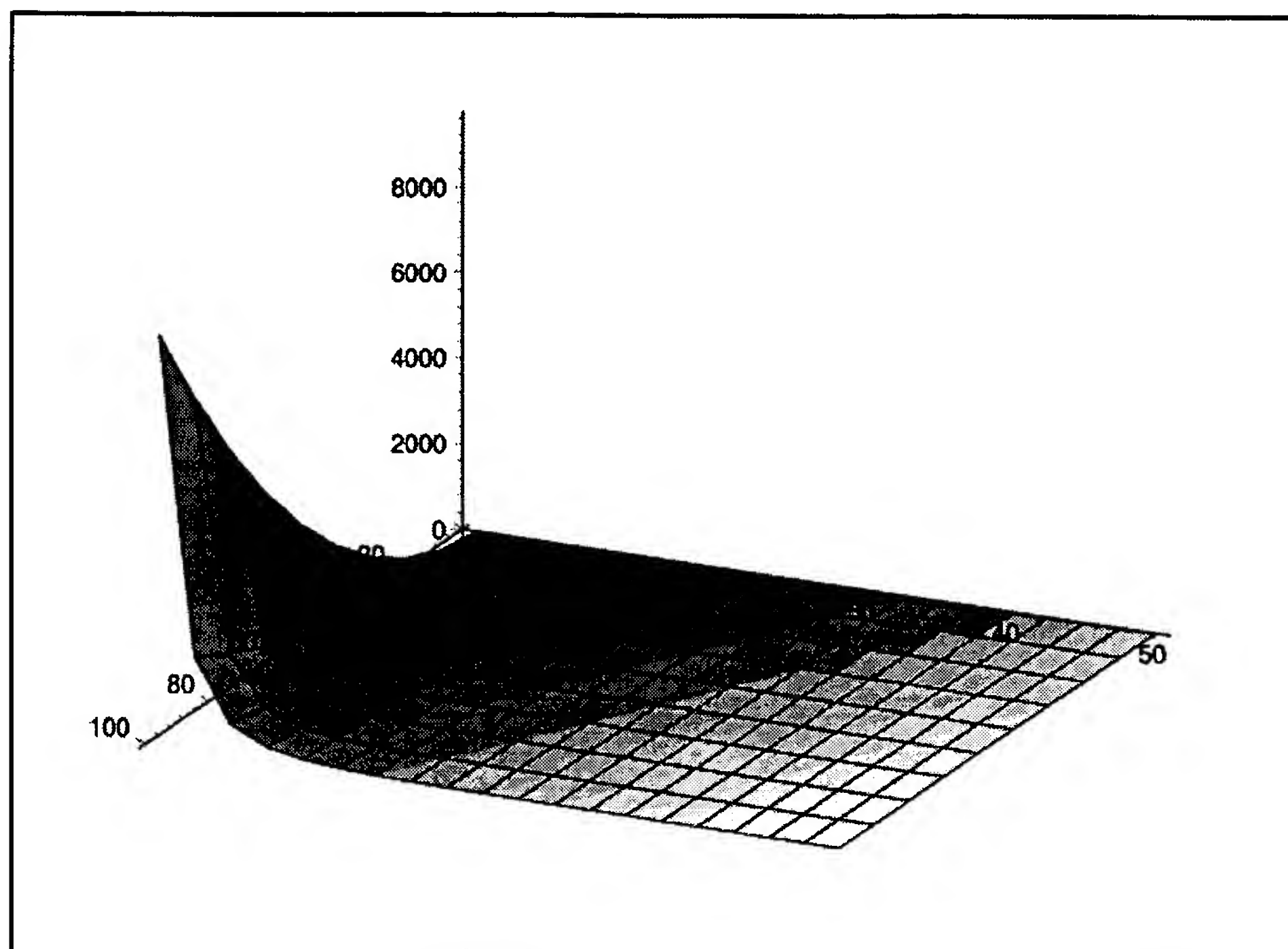
**Figure 16**: $in$-Many Vertex Convex Area Of Radius $r$

to be the optimal number of SDU's, where $z = \mu$ satisfies[16]

$$A(z - \pi) + \pi r^2 \sin z = 0$$

## 57. MAPLE Implementation Code

```
> X:=(n/2)*r^2*sin(Pi*(1-(2/n)));
```

$$X := \frac{1}{2} n r^2 \sin(\pi (1 - \frac{2}{n}))$$

```
> solve(A=X,r)[1];
```

$$\frac{\sqrt{2}\sqrt{n \sin(\dfrac{\pi(n-2)}{n}) A}}{n \sin(\dfrac{\pi(n-2)}{n})}$$

```
> subs(A=1000,n=9,A=X);
```

$$1000 = \frac{9}{2} r^2 \sin(\frac{7\pi}{9})$$

```
> fsolve(%,r,0..100);
```

$$18.59345062$$

---

[16] Note that if $A(\mu - \pi) + \pi r^2 \sin \mu = 0$, then

$$
\begin{aligned}
\frac{n}{2} r^2 \sin\left(\pi\left(1 - \frac{2}{n}\right)\right)\Big|_{n = \frac{2}{r^2} A \csc \mu} 
&= \frac{\frac{2}{r^2} A \csc \mu}{2} r^2 \sin\left(\pi\left(1 - \frac{2}{\frac{2}{r^2} A \csc \mu}\right)\right) \\
&= A \csc \mu \sin\left(\pi\left(1 - \frac{r^2}{A \csc \mu}\right)\right) \\
&= A \csc \mu \sin\left(\left(\pi - \frac{1}{A}\pi r^2 \sin \mu\right)\right) \\
&= A \csc \mu \sin\left(\left(\pi + (\mu - \pi)\right)\right) \\
&= A \csc \mu \sin \mu \\
&= A
\end{aligned}
$$

```
> solve(A=X,n);
```

$$\frac{2A}{r^2 \sin(\text{RootOf}(A\_Z + \pi r^2 \sin(\_Z) - A\pi))}$$

```
> subs(A=1000,r=18.25,A=X);
```

$$1000 = 166.5312500\, n \sin(\pi\,(1 - \frac{2}{n}))$$

```
> fsolve(%,n,4..100);
```

$$12.10617840$$

```
> plot3d(X,r=0..10,n=3..20,
> orientation=[55,60],axes=normal,grid=[10,20]
> );
```
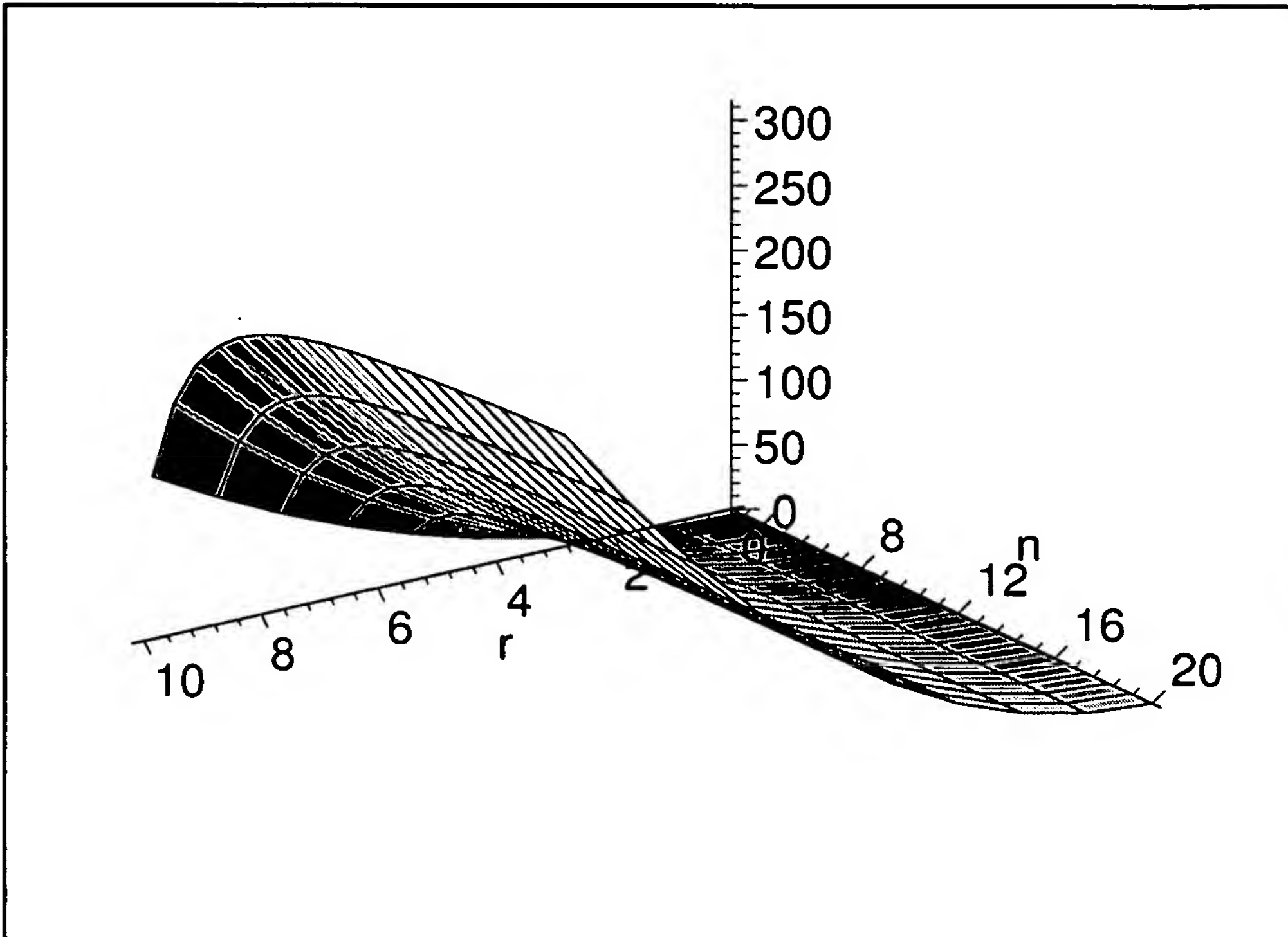


```
> dXdn:=diff(X,n);
```

$$dXdn := \frac{1}{2} r^2 \sin(\pi\,(1 - \frac{2}{n})) + \frac{r^2 \cos(\pi\,(1 - \frac{2}{n}))\,\pi}{n}$$

```
> plot3d(dXdn,r=0..100,n=3..50,
> orientation=[25,60],axes=normal,grid=[10
> ,20]);
```

```
> C:=subs(r=25,dXdn);
```

$$C := \frac{625}{2} \sin(\pi\,(1 - \frac{2}{n})) + \frac{625 \cos(\pi\,(1 - \frac{2}{n}))\,\pi}{n}$$

```
> solve(C=10,n);
```

$$-2\pi/(\text{RootOf}((15625\sin(\_Z)^2 - 15625)\,\_Z^2 + (-31250\,\pi\sin(\_Z)^2 + 31250\,\pi)\,\_Z$$
$$+ 15625\,\pi^2 \sin(\_Z)^2 - 15625\,\pi^2 + 15625\sin(\_Z)^2 - 1000\sin(\_Z) + 16) - \pi)$$

**58.** ⟨Coverage Area Optimization 58⟩ ≡          /* Embedded In MAPLE Code */
This code is used in chunk 12.

## 59. Sample Calculations And Results

```
>  fsolve(C=10,n=3..50);
```
$$13.62494565$$
```
>  fsolve(C=5,n=3..50);
```
$$17.21207895$$
```
>  fsolve(C=1,n=3..50);
```
$$29.51887628$$

# The Index

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⋄ Timothy Hall
PQI Consulting
P. O. Box 425616
Cambridge, MA 02142-0012
osmmts@pqic.com
http://www.pqic.com/osmmts